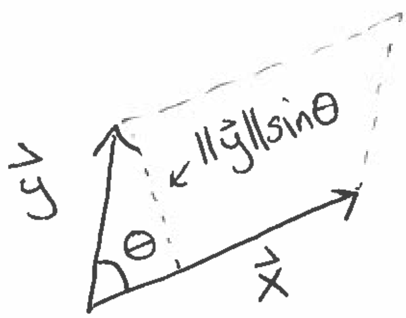
Determinants and the Cross Product.

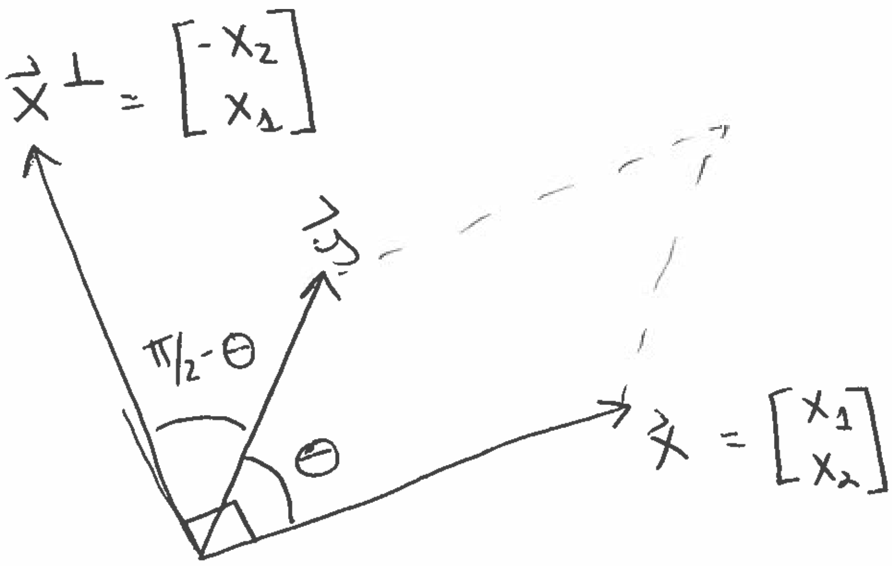
We are now going to introduce a new (scalar) quantity associated to a square matrix.



The parallelogram created by $\vec{x},\vec{y}$ has base $\|\vec{x}\|$ and height $\|\vec{y}\|\sin\theta$, so

$$\text{area}(\vec{x},\vec{y}) = \|\vec{x}\|\,\|\vec{y}\|\sin\theta$$

If $\sin\theta < 0$, we will say that the area is negative, so this is really "signed area".



Now we know that

$$\sin\theta = \cos(\pi/2 - \theta)$$

If we define

$$\vec{x}^{\perp} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

then we'll see that $\vec{x}^{\perp}$ is the rotation of $\vec{x}$ ccw by $\pi/2$. So

$$\begin{aligned} \vec{x}^{\perp} \cdot \vec{y} &= \|\vec{x}^{\perp}\| \, \|\vec{y}\| \cos(\pi/2 - \theta) \\ &= \|\vec{x}\| \, \|\vec{y}\| \sin\theta \end{aligned}$$

This gives us an explicit formula

$$\text{area}(\vec{x},\vec{y}) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1y_2 - x_2y_1$$

~~We are~~

Definition. The <u>determinant</u> of a 2x2 matrix $\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$ is given by

$$\operatorname{Det}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = x_1y_2 - x_2y_1.$$

Note. This is also written $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$ or $\mathcal{D}(\vec{x},\vec{y})$.

We will use

$$\operatorname{Det} A = \operatorname{Det}\begin{bmatrix} \uparrow & & \uparrow \\ A_1 & \cdots & A_n \\ \downarrow & & \downarrow \end{bmatrix}$$

$$= \operatorname{Det}(\vec{A}_1, \dots, \vec{A}_n)$$

Proposition. For $\vec{A} = [\vec{x}, \vec{y}] \in Mat_{2\times 2}$,

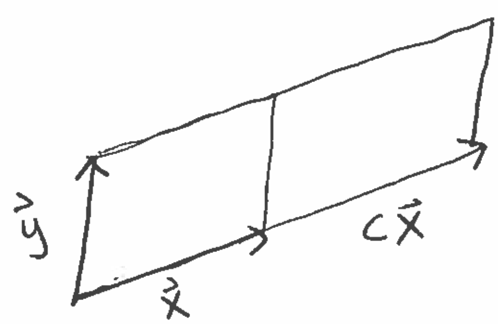
1) $Det(\vec{x}, \vec{y}) = -Det(\vec{y}, \vec{x})$
2) $Det(c\vec{x}, \vec{y}) = Det(\vec{x}, c\vec{y}) = c\,Det(\vec{x}, \vec{y})$
3) $Det(\vec{x} + \vec{z}, \vec{y}) = Det(\vec{x}, \vec{y}) + Det(\vec{z}, \vec{y})$

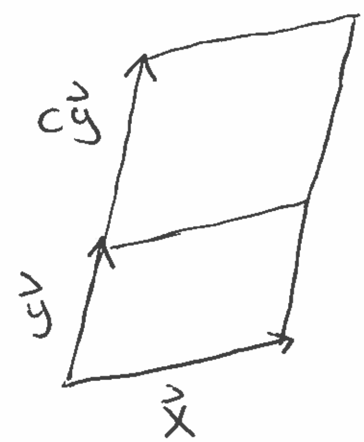
   $Det(\vec{x}, \vec{z} + \vec{y}) = D(\vec{x}, \vec{y}) + Det(\vec{x}, \vec{z})$
4) $Det(\vec{e}_1, \vec{e}_2) = 1.$

Proof. We will prove 1 algebraically:

$$Det(\vec{x}, \vec{y}) = x_1 y_2 - x_2 y_1$$
$$Det(\vec{y}, \vec{x}) = y_1 x_2 - y_2 x_1$$
$$\Rightarrow Det(\vec{x}, \vec{y}) = -Det(\vec{y}, \vec{x}).$$

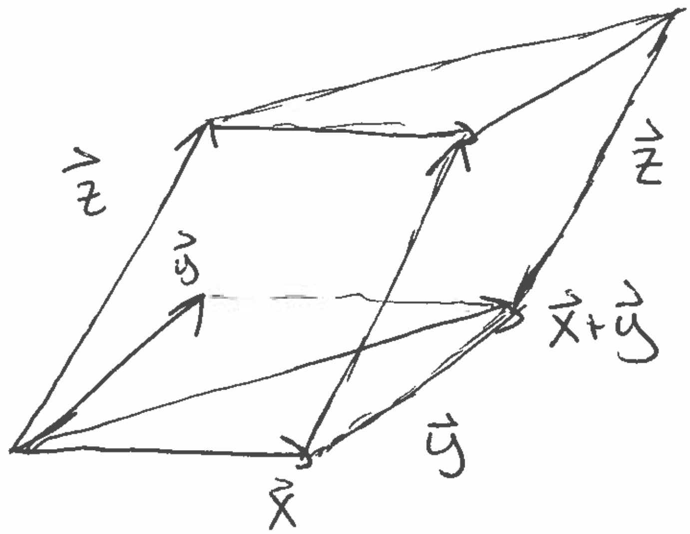
For 2,



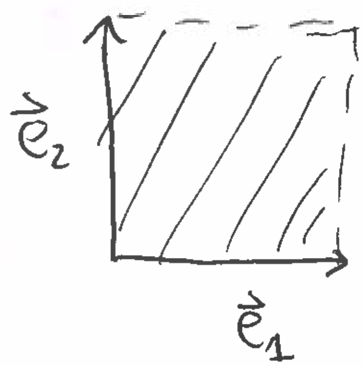


area increases by a factor of c when we scale base <u>or</u> height.

For 3,



For 4,



area of unit square is 1.

$\square$

We now want to extend the definition of determinant to more vectors in more dimensions, keeping the idea "determinant is signed volume".

Definition. For $A = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{x} & \vec{y} & \vec{z} \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \in \mathrm{Mat}_{3\times 3}$,

we define

$$\mathrm{Det}(\vec{x},\vec{y},\vec{z}) = x_1 \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} - x_2 \begin{vmatrix} y_1 & z_1 \\ y_3 & z_3 \end{vmatrix} + x_3 \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}$$

2x2 determinants

With this definition, $\mathrm{Det}(\vec{x},\vec{y},\vec{z})$ is much like $\mathrm{Det}(\vec{x},\vec{y})\vec{z}$.

Proposition.

1) Det changes sign when two columns are interchanged (alternating)

2) Det is linear in each column (multilinear)

We will later see that these properties uniquely <u>define</u> Det.

Definition. The <u>cross product</u> of $\vec{x}, \vec{y} \in \mathbb{R}^3$ is given by

$$\vec{x} \times \vec{y} = (x_2 y_3 - x_3 y_2,\ x_3 y_1 - x_1 y_3,\ x_1 y_2 - x_2 y_1).$$

Properties of cross.