Differentiability.

Definition. \( f'(a) = \lim_{h \to 0} \frac{f(\alpha h) - f(\alpha)}{h} \), if this limit exists.

Redefinition. \( f'(a) = m \iff \lim_{h \to 0} \frac{f(\alpha h) - f(\alpha) - mh}{h} = 0 \)

Prove these are same.

\[
\text{Error} = \frac{f(\alpha h) - f(\alpha)}{h} = f'(\alpha) h
\]

The derivative makes \( \lim_{h \to 0} \frac{\text{Error}(h)}{h} = 0 \).

Definition. Let \( U \subset \mathbb{R}^n \) be open and \( \alpha \in U \). A function \( \hat{f}: U \to \mathbb{R}^m \) is differentiable at \( \alpha \) if there exists a linear map \( D\hat{f}(\alpha) : \mathbb{R}^n \to \mathbb{R}^m \) so

\[
\lim_{h \to 0} \frac{\hat{f}(\alpha + h) - \hat{f}(\alpha) - (D\hat{f}(\alpha))(h)}{\|h\|} = 0
\]

This linear map is called the differential or Jacobian.

Definition. The tangent plane (or tangent space) of \( \hat{f}(\alpha) \) at \( \alpha \) is the graph of \( g(x) = \hat{f}(\alpha) + D\hat{f}(\alpha)(x-\alpha) \).

Proposition. If \( \hat{f}: \mathbb{R}^n \to \mathbb{R}^m \) is differentiable at \( \alpha \) then the partial derivatives \( \frac{\partial \hat{f}}{\partial x_i} \) exist and

\[
D\hat{f}(\alpha) = \begin{bmatrix} \frac{\partial \hat{f}}{\partial x_1}(\alpha) \\
\vdots \\
\frac{\partial \hat{f}}{\partial x_m}(\alpha) \end{bmatrix}
\]