Linear Transformations (continued)

We define

<u>Definition</u>. The identity <u>matrix</u> I_n is the $n\times n$ diagonal matrix with 1's along the diagonal.

<u>Definition</u>. The identity <u>map</u> $id_n : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $id(\vec{v}) = \vec{v}$ for all $\vec{v} \in \mathbb{R}^n$.

Lemma. $I_n = [id_n]$.

Proposition. If $A, A' \in Mat_{m\times n}$, $B, B' \in Mat_{n\times p}$, $C \in Mat_{p\times q}$ and $c \in \mathbb{R}$,

1. $AI_n = A = I_m A$

2. $(A+A')B = AB + A'B$

$A(B+B') = AB + AB'$

3. $(cA)B = c(AB) = A(cB)$

4. $(AB)C = A(BC)$.

Proof. If we think about matrix multiplicatn as composition of the associated linear transformations, all of these are obvious except for 2.

If $A=[S]$, $A'=[S']$, $B=[T]$,

$$((S+S')\circ T)(\vec{v})$$
$$= (S+S')(T(\vec{v})) = S(T(\vec{v})) + S'(T(\vec{v}))$$
$$= (S\circ T)(\vec{v}) + (S'\circ T)(\vec{v})$$

$\square$

Definition. Let A be an $n \times n$ matrix. We say A is <u>invertible</u> if there exists an $n \times n$ matrix A^{-1} so that

$$AA^{-1} = A^{-1}A = I_n.$$

Note. If $A = [T]$ for a linear transform. $T: \mathbb{R}^n \to \mathbb{R}^n$, then $A^{-1} = [T^{-1}]$ where T^{-1} is defined to be the map $\mathbb{R}^n \to \mathbb{R}^n$ so that $T(T^{-1}(\vec{v})) = \vec{v}$ and $T^{-1}(T(\vec{v})) = \vec{v}$.

Example. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

We prove this by multiplying

$$AA^{-1} = \frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & \cancel{-ba+ab} \\ \cancel{cd-cd} & -bc+ad \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

Proposition. Suppose A, B are invertible $n \times n$ matrices. Then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof. This is an easy computation:

$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= AI_nA^{-1} \\ &= AA^{-1} \\ &= I_n. \qquad \blacksquare\end{aligned}$$

Checking $(B^{-1}A^{-1})(AB) = I_n$ is similar. $\square$

~~Proposition. (Matrix inverses are unique)~~
~~If $AB = I_n$ and $AC = I_n$ then~~
~~$B = C = A^{-1}$.~~

~~Proof. $AB = I_n = AA^{-1}$ so $A^{-1}AB = A^{-1}AA^{-1}$,~~
~~and $I_nB = I_nA^{-1}$ and~~

Proposition. (Matrix inverses are unique.)
Suppose $AB = BA = I$ and $AC = CA = I$.
Then $B = C$.

Proof. $AB = I$, so $AB = AC$. Multiplying by C on the left,

$$C(AB) = C(AC)$$

$$(CA)B = (CA)C$$

$$IB = IC$$

$$B = C.$$

$\square$

Note. Even if A is a 1×1 matrix $A = [a_{11}]$, we have seen "invertible numbers" and "non-invertible numbers".