The derivative.

Definition. If $U \subset \mathbb{R}^n$ is open, $\vec{f}: U \to \mathbb{R}^m$, the <u>partial</u> <u>derivative</u>

$$\frac{\partial \vec{f}}{\partial x_j}(\vec{a}) = \lim_{t \to 0} \frac{\vec{f}(\vec{a} + t\vec{e}_j) - \vec{f}(\vec{a})}{t}$$

(if this limit exists).

Notations. $D_{\vec{e}_j}\vec{f}$, $\vec{f}_{x_j}$, $D_j\vec{f}$ (according to Shifrin)

Idea: $\frac{\partial \vec{f}}{\partial x_j}$ is the derivative w.r.t. x_j treating all other variables as constants.

Example. $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x \sin y$.

Definition. If $U \subset \mathbb{R}^n$ is open, $\vec{f}: U \to \mathbb{R}^m$, the <u>directional</u> <u>derivative</u> of $\vec{f}$ in the direction $\vec{v} \in \mathbb{R}^n$

$$D_{\vec{v}} f(\vec{a}) = \lim_{t \to 0} \frac{\vec{f}(\vec{a} + t\vec{v}) - f(\vec{a})}{t}$$

(if this limit exists).

Idea. $D_{\vec{v}}\vec{f}$ is the rate of change of $\vec{f}$ experienced by an observer moving with velocity $\vec{v}$ at point $\vec{a}$.

Example. $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 y + e^{2x-y}$, $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

Compute $D_{\vec{v}} f(\vec{a})$ by defining $\varphi(t) = f(\vec{a} + t\vec{v})$ and observing that $D_{\vec{v}} f(\vec{a}) = \varphi'(0)$.

Example. $f(\vec{x}) = \|\vec{x}\|$. Let $\vec{v} = \dfrac{\vec{a}}{\|\vec{a}\|}$. $D_{\vec{v}} f(\vec{a})$?

~~So~~ Compute limit directly.

Example. $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \dfrac{|x|y}{\sqrt{x^2+y^2}}$. If $\|\vec{v}\| = 1$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$,

compute $D_{\vec{v}} f(\vec{0}) = |v_1| v_2$.