The derivative.

Definition. If \( \mathbb{R}^n \rightarrow \mathbb{R}^n \) is open, the partial derivative

\[
\frac{\partial \tilde{f}}{\partial x_j}(\tilde{a}) = \lim_{t \to 0} \frac{\tilde{f}(\tilde{a} + t\tilde{e}_j) - \tilde{f}(\tilde{a})}{t}
\]

(if this limit exists).

Notations. \( \tilde{f} \), \( \tilde{f}_{x_j} \), \( D_j \tilde{f} \) (according to Shifrin).

Idea: \( \frac{\partial \tilde{f}}{\partial x_j} \) is the derivative w.r.t. \( x_j \) treating all other variables as constants.

Example. \( f([x, y]) = x \sin y \).

Definition. If \( \mathbb{R}^n \rightarrow \mathbb{R}^m \) is open, \( \tilde{f} : U \rightarrow \mathbb{R}^m \), the directional derivative of \( \tilde{f} \) in the direction \( \tilde{v} \in \mathbb{R}^n \)

\[
D_{\tilde{v}} \tilde{f}(\tilde{a}) = \lim_{t \to 0} \frac{\tilde{f}(\tilde{a} + t\tilde{v}) - \tilde{f}(\tilde{a})}{t}
\]

(if this limit exists).

Idea. \( D_{\tilde{v}} \tilde{f} \) is the rate of change of \( \tilde{f} \) experienced by an observer moving with velocity \( \tilde{v} \) at point \( \tilde{a} \).
Example. \( f([x\ y]) = x^2 y + e^{2x-y} \quad \hat{a} = [\frac{1}{2}], \quad \hat{v} = [-\frac{1}{3}] \).

Compute \( D_{\hat{v}} f(\hat{a}) \) by defining \( \Phi(t) = f(\hat{a} + t\hat{v}) \) and observing that \( D_{\hat{v}} f(\hat{a}) = \Phi'(0) \).

Example. \( f(\hat{x}) = \| \hat{x} \| \). Let \( \hat{v} = \frac{\hat{a}}{\| \hat{a} \|} \). \( D_{\hat{v}} f(\hat{a}) \)?

So compute limit directly.

Example. \( f([x\ y]) = \frac{1 x y}{\sqrt{x^2 + y^2}} \). If \( \| \hat{v} \| = 1 \), \( \hat{v} = [v_1\ v_2] \),

compute \( D_{\hat{v}} f(\hat{a}) = 1 v_1 | v_2 | \).