Quadratic forms and the second derivative test

Recall.

If $f''$ is $C^2$ in $(a-r, a+r)$ for $r>0$ and $f'(a)=0$, then

- $f''(a) > 0 \Rightarrow a$ is a local min
- $f''(a) < 0 \Rightarrow a$ is a local max
- ($f''(a) = 0$, doesn’t rule out anything)

Why?

Lemma. Suppose $g: [0, 1] \to \mathbb{R}$ is $C^2$. Then

$$g(1) = g(0) + g'(0) + \frac{1}{2} g''(\xi)$$

for some $\xi \in (0, 1)$.

Proof. (Clever!) Let

$$P(t) = g(0) + g'(0)t + Ct^2$$
observe that if

$$C = g(1) - g(0) - g'(0)$$

we have

$$P(1) = g(1)$$
$$P(0) = g(0)$$
$$P'(0) = g'(0)$$

So consider \( h(t) = g(t) - TP(t) \). We have

$$h(0) = 0, \ h(1) = 0$$

so (Rolle's theorem) \( \exists \) some \( c \in (0, 1) \) so that \( h'(c) = 0 \). Now

$$h'(0) = 0, \ h'(c) = 0$$

so (Rolle's theorem) \( \exists \) some \( \xi \in (0, c) \) so that \( h''(\xi) = 0 \).
Now
\[ D = h''(\xi) = g''(\xi) - p''(\xi) \]
\[ = g''(\xi) - 2C \]
so \( g''(\xi) = 2C \). But then
\[ P(t) = g(0) + g'(0) t + \frac{t^2}{2} g''(\xi) \]
and \( P(1) = g(1) = g(0) + g'(0) \xi + \frac{1}{2} g''(\xi) \).

This is the proof of the 2nd derivative test: If \( f''(a) > 0 \), then (by continuity) \( \exists \) some \( \varepsilon > 0 \) such that \( f''(x) > 0 \) on \( (a-\varepsilon, a+\varepsilon) \).

At any \( y \in (a, a+\varepsilon) \), \( \exists \) some \( \xi \in (a, y) \) so that
\[ f(y) = f(a) + f'(a)(y-a) + \frac{1}{2} f''(\xi)(y-a)^2 \]
= f(a) + o + \frac{1}{2} f''(\xi)(y-a)^2 > 0

so f(y) > f(a).

We now want to generalize.


Prop. Suppose \( f: B(a,r) \rightarrow \mathbb{R} \) is \( C^2 \), for \( 1/2 \| h \| < r \),

\[ f(a+h) = f(a) + Df(a)h + \frac{1}{2} h^T \text{Hess}(a)h \]

for some \( 0 < \xi < 1 \).

Definition. Pos def, neg def, psd, nsd, ind

Example. \[
\begin{bmatrix}
1 & 2 \\
2 & 5
\end{bmatrix}, \quad
\begin{bmatrix}
4 & 1 \\
1 & -1
\end{bmatrix}
\]

Prop. Suppose \( f: B(a,r) \rightarrow \mathbb{R} \) is \( C^2 \), \( Df(a) = 0 \)

pos def \( \rightarrow \) min, neg def \( \rightarrow \) max

indef \( \rightarrow \) saddle, (psd or nsd tells us)

nothing
\[ AD - B^2 > 0 \quad A > 0 \quad \text{local min} \]
\[ AD - B^2 < 0 \quad A < 0 \quad \text{local max} \]
\[ AD - B^2 < 0 \quad \text{saddle} \]
\[ (AD - B^2 = 0 \text{ tells us nothing}) \]

Example: \[ x^3 + y^2 - 6xy \]

LU decomp.  LDL^T decomp.

not more to come