Topology in $\mathbb{R}^n$

Definition. Let $\vec{a} \in \mathbb{R}^n$ and $\delta > 0$. The **ball of radius δ centered at $\vec{a}$** is

$$B(\vec{a}, \delta) = \{ \vec{x} \in \mathbb{R}^n \mid \|\vec{x} - \vec{a}\| < \delta \}.$$

If we have $[a_1, b_1], \dots, [a_n, b_n]$, then the **rectangle**

$$R = [a_1, b_1] \times \dots \times [a_n, b_n]$$
$$= \{ \vec{x} \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i \text{ for } i \in 1, \dots, n \}$$

If we have $(a_1, b_1), \dots, (a_n, b_n)$, then the **open rectangle**

$$S = (a_1, b_1) \times \dots \times (a_n, b_n)$$
$$= \{ \vec{x} \in \mathbb{R}^n \mid a_i < x_i < b_i \text{ for } i \in 1, \dots, n \}$$

Proposition. If $\vec{a} \in \mathbb{R}^n$ and $\delta > 0$, then

$$\left[a_1 - \frac{\delta}{\sqrt{n}}, a_1 + \frac{\delta}{\sqrt{n}}\right] \times \dots \times \left[a_n - \frac{\delta}{\sqrt{n}}, a_n + \frac{\delta}{\sqrt{n}}\right] \subset B(\vec{a}, \delta)$$

$$B(\vec{a}, \delta) \subset [a_1 - \delta, a_1 + \delta] \times \dots \times [a_n - \delta, a_n + \delta].$$

Proof. Let's call the first rectangle S.

Then

$$\begin{aligned}
\vec{x} \in S \quad & \Rightarrow \quad a_i - \frac{\delta}{\sqrt{n}} < x_i < a_i + \frac{\delta}{\sqrt{n}} \text{ for all } i \in 1, \dots, n \\
& \Rightarrow \quad |x_i - a_i| < \frac{\delta}{\sqrt{n}} \text{ for all } i \in 1, \dots, n \\
& \Rightarrow \quad (x_i - a_i)^2 < \frac{\delta^2}{n} \text{ for all } i \in 1, \dots, n \\
& \Rightarrow \quad \sum_{i=1}^{n} (x_i - a_i)^2 < \delta^2 \\
& \Rightarrow \quad \|\vec{x} - \vec{a}\| < \delta.
\end{aligned}$$

Let's call the second rectangle R.

Then

$$\vec{x} \in B(\vec{a}, \delta) \Rightarrow \|\vec{x} - \vec{a}\| < \delta$$

$$\Rightarrow \sqrt{\sum_i (x_i - a_i)^2} < \delta$$

$$\Rightarrow \sum_i (x_i - a_i)^2 < \delta^2$$

$$\Rightarrow (x_j - a_j)^2 \leq \delta^2 \text{ for each } j \in 1, \ldots, n$$

because $(x_j - a_j)^2 \leq \sum_{i=1}^{n} (x_i - a_i)^2$

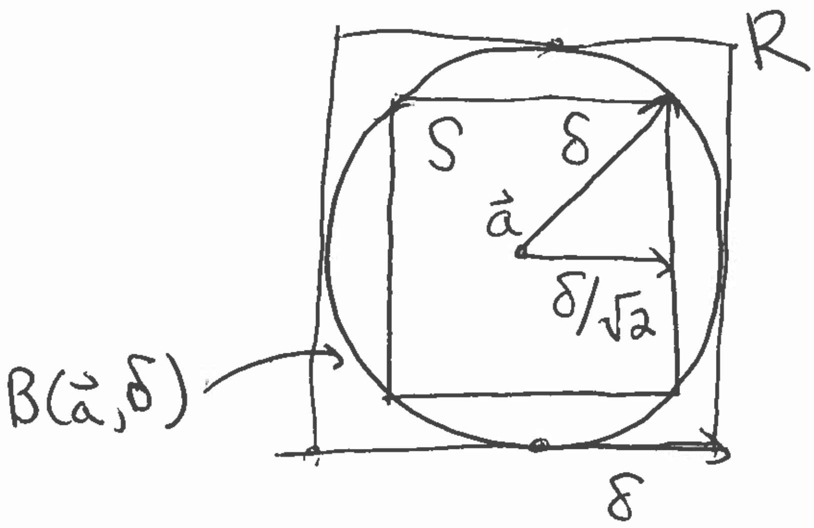
$$\Rightarrow |x_j - a_j| \leq \delta \text{ for all } j \in 1, \ldots, n$$

$$\Rightarrow a_j - \delta \leq x_j \leq a_j + \delta \text{ for all } j \in 1, \ldots, n$$

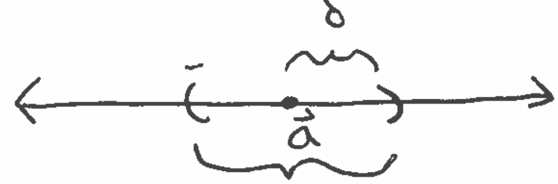
$$\Rightarrow \vec{x} \in R.$$

$\square$

In $\mathbb{R}^2$,



In $\mathbb{R}^1$,



$B(\vec{a},\delta) = S = (\vec{a}-\delta, \vec{a}+\delta)$

$R = [a-\delta, a+\delta]$.

swap order in class.

Definition. We say $U \subset \mathbb{R}^n$ is <u>open</u> if for every $\vec{a} \in U$ there is some $\delta_{\vec{a}}$ so that $B(\vec{a}, \delta_{\vec{a}}) \subset U$.

Idea. In $\mathbb{R}$, open intevals (a,b) are open sets. So are ~~this~~ unions of open intervals...

Example.

$$U = \{ \vec{x} \in \mathbb{R}^2 \mid 0 < x_1 x_2 < 1 \}$$ is open.

Proof. "Wlogwma" we have chosen some $\vec{a} \in \begin{bmatrix} a \\ b \end{bmatrix}$ with $a, b > 0$. Since

$$ab < 1$$

there is some $\epsilon > 0$ so that

$$ab(1+\epsilon)^2 < 1$$

or

$$a(1+\epsilon)\, b(1+\epsilon) < 1$$

or

$$(a + \epsilon a)(b + \epsilon b) < 1.$$

We will choose $\delta < \min(\epsilon a, \epsilon b)$.

On the other side, since

$$ab > 0$$

if we choose $\delta < \min(a, b)$ we have

$$(a-\delta)(b-\delta) > 0$$

because both are positive. Thus for $\delta < \min(\epsilon a, \epsilon b, a, b)$, we have

$$R = [a-\delta, a+\delta] \times [b-\delta, b+\delta] \subset \mathcal{U}.$$

This rectangle contains the ball

$$B(\vec{a}, \delta) \subset R \subset \mathcal{U},$$

so for every $\vec{a} \in \mathcal{U}$ there is some δ with $B(\vec{a}, \delta) \subset \mathcal{U}$ and $\mathcal{U}$ is open. $\square$

Definition. A <u>sequence</u> $\{\vec{x}_k\} \in \mathbb{R}^n$ is an infinite list $\vec{x}_1, \ldots, \vec{x}_k, \ldots$

The sequence $\{\vec{x}_k\}$ converges to $\vec{a}$ if for all $\epsilon > 0$ there is some $K \in \mathbb{N}$ so that

$$\|\vec{x}_k - \vec{a}\| < \varepsilon \text{ for all } k > K.$$

In this case, we write

$$\lim_{k \to \infty} \vec{x}_k = \vec{a}$$

Examples. Let $\vec{x}_0$ be any vector in $\mathbb{R}^n$ and let $\vec{x}_k = \frac{1}{2}\vec{x}_{k-1}$ define $\{\vec{x}_k\}$. We claim

$$\lim_{k \to \infty} \vec{x}_k = \vec{0} \text{ for all choices of } \vec{x}_0.$$