Day 1. Math 3500.

Intro and mechanics. (10 min)

Integrated multivariable calc/linear algebra.
Advanced course, depends on reading/homework.

Read → Class (discuss harder → Homework
points from section
applications)

Gradescope. JB22BP.
Bookings.

Diagnostic/Reading Quiz. (≤10 min)

Definition. A vector \( \mathbf{x} \in \mathbb{R}^n \) is an ordered list of numbers

\[
\mathbf{x} = \begin{bmatrix}
    x_1 \\
    \vdots \\
    x_n
\end{bmatrix}
\]

We can think of these as coordinates of a
point in an $n$-dimensional space, and
the vector as an arrow from the
origin $\mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ to $\mathbf{X}$.

Example.

\[
\mathbf{X} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}
\]

Definition. We define the length of $\mathbf{X}$
to be $\|\mathbf{X}\| = \sqrt{x_1^2 + \ldots + x_n^2}$, and say that
$\mathbf{X}$ is a unit vector if $\|\mathbf{X}\|=1$.

Note. This definition implies the Pythagorean
theorem. It's not the only useful definition
of length for vectors.
Definition. If $c \in \mathbb{R}$ and $x \in \mathbb{R}^n$, we say $c$ is a scalar and define the scalar multiplication of $c$ and $x$ to be

$$c x = \begin{bmatrix} cX_1 \\ \vdots \\ cX_n \end{bmatrix}$$

Example.

$$2x = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\frac{x}{\|x\|} = \frac{1}{\|x\|} x = \frac{1}{\sqrt{4+9}} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{bmatrix}$$

$$-x = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

Definition. $x$ and $y \in \mathbb{R}^n$ are parallel if there exists some $c$ such that $x = cy$. 
Definition. If \( x, y \in \mathbb{R}^n \) we define
\[
    x + y = \begin{bmatrix}
    x_1 + y_1 \\
    \vdots \\
    x_n + y_n
\end{bmatrix}.
\]

Example.

We call this the "parallelogram law" for vector addition. We add vectors by placing them "tail to head."
Lemma,

\[ x + (-1)y = x - y = \begin{bmatrix} x_1 - y_1 \\ \vdots \\ x_n - y_n \end{bmatrix} \]

Why were vectors first important? Geometry started with axioms from which propositions and theorems were derived by logical deduction. A great deal was accomplished this way, but the arguments require insight and creativity—they are also very long (the Pythagorean theorem is the 47th proposition in Euclid). Vectors and coordinate geometry make many geometric proofs almost mechanical exercises in vector algebra.
If we think about

we see that the midpoint of the line segment is the point halfway from \( x \) to \( y \), given by

\[
x + \frac{1}{2}(y-x) = \frac{1}{2}y + \frac{1}{2}x
\]

\[
= \frac{1}{2}(x+y)
\]

\[
= m
\]
Proposition: The diagonals of a parallelogram bisect each other.

Proof. We can write the parallelogram as

Now the diagonals join 0 to \( x+y \), with midpoint

\[
m_1 = \frac{1}{2} (0 + (x+y)) = \frac{1}{2} x + \frac{1}{2} y
\]
and $x$ to $y$, with midpoint

$$m_2 = \frac{1}{2}(x+y) = \frac{1}{2}x + \frac{1}{2}y.$$ 

These are the same point. □

Definition. The **median** of $\triangle ABC$ through $A$ is the line segment from $A$ to the midpoint of $BC$. (Other medians similar.)

Proposition. The three medians of $\triangle ABC$ intersect at a single point $P$. 

![Diagram of a triangle with medians intersecting at a single point P]
Proof. We may write $\Delta ABC$ as

\[
\begin{align*}
\frac{1}{2} y \\
\frac{1}{2} (x+y) \\
x
\end{align*}
\]

We now construct the point $\frac{2}{3}$ of the way from each vertex to the opposite side's midpoint.

\[
P_1 = O + \frac{2}{3} \left( \frac{1}{2} (x+y) - 0 \right) = \frac{1}{3} x + \frac{1}{3} y
\]

\[
P_2 = x + \frac{2}{3} \left( \frac{1}{2} y - x \right) = x - \frac{2}{3} x + \frac{1}{3} y = \frac{1}{3} x + \frac{1}{3} y.
\]
\[ P_3 = y + \frac{2}{3} \left( \frac{1}{2} x - y \right) \]

\[ = y - \frac{2}{3} y + \frac{1}{3} x = \frac{1}{3} x + \frac{1}{3} y. \]

Since these are all the same point \( p \), we have completed the proof. \( \square \)