Math 4250/6250 Exam #1

This take-home exam covers the material on curves (class notes #1 through #6 and DoCarmo Chapter #1). Please pick 4 of the following problems. If you are a graduate student, you must attempt the challenge problem. If you are an undergraduate, you can do the challenge problem as one of your four problems (and will get bonus credit if you get it right).

You **are** permitted to use	You **are not** permitted to use
Maple (or Mathematica or MATLAB)	The internet
A calculator (or graphing calculator)	
DoCarmo	Other books
Your notes	Other people's notes
Your brain	Other people's brains
Class notes posted on the website	

1. Find the curvature and torsion of the following curves
   (1) $\alpha(t) = \left(t, \frac{1+t}{t}, \frac{1-t^2}{t}\right), t > 0.$
   (2) $\beta(t) = \left(a(3t - t^3), 3at^2, a(3t + t^3)\right).$
   Please notice that **neither curve is parametrized by arclength**. Do you notice anything unusual about either curve? (Look again after you do Problem 4.)
2. Suppose that $\alpha(s) : [0, \ell] \to \mathbf{R}^3$ is a regular curve parametrized by arclength. The *tangent indicatrix* of α is the curve $T(s)$ on the unit sphere S^2. We proved in class that the length of the tangent indicatrix is the total curvature of α, or

$$\text{Length}\, T(s) = \int_0^\ell \kappa(s)\, \mathrm{d}s. \tag{1}$$

   The *normal indicatrix* is the curve $N(s)$ on the sphere S^2, while the *binormal indicatrix* is the curve $B(s)$ on the sphere S^2. Find formulas for

$$\text{Length}\, N(s) \quad \text{and} \quad \text{Length}\, B(s)$$

   similar to Equation (1) above. (Keep in mind that the length of the *vectors* $N(s)$ and $B(s)$ are always one for any value of s, since these are unit vectors. When we write length here, we're thinking about the length of the curves traced out by these vectors on the unit sphere as s increases.)
3. Let k, k', and t be real-valued constants with $k \geq 0$. Prove that the cubic polynomial curve

$$\alpha(s) = \left(s - \frac{k^2}{6}s^3, \frac{k}{2}s^2 + \frac{k'}{6}s^3, -\frac{kt}{6}s^3\right)$$

   has curvature $\kappa(0) = k$, derivative of curvature $\kappa'(0) = k'$, and torsion $\tau(0) = t$. (Be careful– α is not arclength parametrized.)

   Any curve can be approximated to various degrees of precision by polynomial curves of varying degree– for instance, the tangent line to $\alpha(s)$ at $s = 0$ is the best linear approximation to α near $\alpha(0)$.

Suppose that $\beta(s)$ is an arclength parametrized curve with curvature $\kappa_\beta(0) = k$, derivative of curvature $\kappa'_\beta(0) = k'$ and torsion $\tau_\beta(0) = t$. Further, suppose that $\beta(0) = \alpha(0)$, $\beta'(0) = \alpha'(0)$ and $\beta''(0) = \alpha''(0)$.

Explain why α is the best cubic approximation to β.

4. A **generalized helix** is a curve whose tangent $T(s)$ makes a constant angle with a line ℓ. Show that a curve $\alpha(s)$ is a generalized helix if and only if

$$\alpha^{(4)}(s) \times \alpha^{(3)}(s) \cdot \alpha^{(2)}(s) = \pm\kappa^5(s)\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\tau(s)}{\kappa(s)}\right) = 0.$$

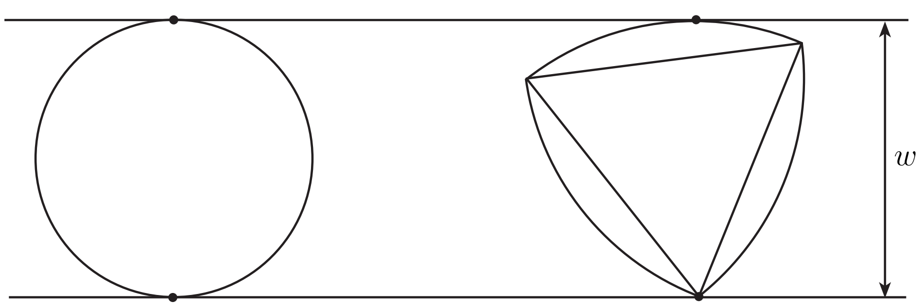
Here $\alpha^{(n)}(s)$ is the nth derivative of α with respect to s. Notice that this only works for arclength parametrized curves. As usual, κ and τ are curvature and torsion. If $\kappa(s) = 0$, ignore the middle equation– such curves can still be generalized helices, but they will be characterized by having the left-hand side equal zero.

5. **Challenge Problem for Graduate Students.** The *width* $w(\theta)$ of a convex curve α in the plane in the direction θ is the absolute value of the difference between the maximum and minimum values of

$$f_\theta(s) = \alpha(s) \cdot (\cos\theta, \sin\theta)$$

A pair of points at which $f_\theta(s)$ attains its maximum and minimum values are called *opposite* points.

Suppose that α is a curve of constant width (that is, that $w(\theta)$ does not depend on θ). Prove that the sum of radii of curvature at any pair of opposite points is equal to the width.



The illustration above shows two curves of constant width w (the circle and the Releaux triangle) with marked pairs of opposite points. The Releaux triangle is made up of three arcs of circles centered at the three vertices of an equilateral triangle. In each case, we see that the distance w is equal to the sum of the radii of curvature at the marked points (the bottom point on the right-hand curve is a corner point, and so has radius of curvature zero).

Hint 1. Prove that the length of the line connecting two opposite points is the width.

Hint 2. A good way to reparametrize a convex curve $\alpha(s)$ is by the angle its tangent vector $T(s) = (\cos\theta, \sin\theta)$ makes with the x-axis. Reparametrize by θ and work out a nice formula for the curvature of $\alpha(\theta)$ using the derivative rule for inverse functions.