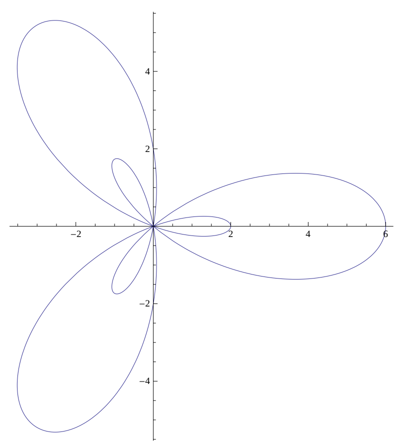
Spring 2012 Math 4250/6250 Exam #1

This take-home exam covers the material on curves (class notes #1 through #6 and DoCarmo Chapter #1). Undergraduates must pick four questions and may substitute one or more graduate exam problems for undergraduate exam problems for bonus credit. Graduate students must pick one undergraduate problem and two graduate problems and may do all three graduate problems for bonus credit.

You **are** permitted to use	You **are not** permitted to use
Maple (or Mathematica or MATLAB)	The internet
A calculator (or graphing calculator)	
DoCarmo	Other books
Your notes	Other people's notes
Your brain	Other people's brains
Class notes posted on the website	

Undergraduate Exam Problems

1. Find the (unsigned) curvature $\kappa(t)$ of the parametrized plane curve given below in *polar* coordinates:



$$\alpha(t) = (r(t), \theta(t)) = (4\cos 3t + 2, t)$$

2. Find the curvature and torsion of the curve given by $\alpha(t) = (1 + \cos t, \sin t, 2\sin \frac{t}{2})$.

3. We learned to write the set of lines in the plane using the coordinates (θ, p) where $\ell(\theta, p)$ is the line which satisfies the equation:

$$(\cos\theta)x + (\sin\theta)y = p,$$

   Suppose that $\alpha(s) = (\alpha_1(s), \alpha_2(s))$ is an arclength-parametrized plane curve, and that $\ell(s)$ is the tangent line to α at $\alpha(s)$. Give formulas $(\theta(s), p(s))$ for the θ, p coordinates of $\ell(s)$ in terms of $\alpha(s)$ and $\alpha'(s)$.

4. Two arclength-parametrized space curves $\alpha(s)$ and $\beta(s)$ have the unusual property that $|\alpha(s) - \beta(s)| = 1$ for all s. Prove or give a counterexample: α and β are either a double helix or are plane curves in parallel planes.

Graduate Exam Problems

1. We saw that you could write the length of a curve in terms of the number of intersections with lines in the plane using the (θ, p) coordinates for lines above as:

$$\mathrm{Length}(\alpha) = \frac{1}{2} \iint I(p, \theta)\, \mathrm{d}p\, \mathrm{d}\theta$$

where $I(p, \theta)$ is the number of intersections between α and $\ell(p, \theta)$. The proof had two key steps:

- Prove the area integral $\mathrm{d}p\, \mathrm{d}\theta$ coordinates was unchanged by a rigid motion of the plane. This means that the integral over any line segment of the same length L was equal.
- Prove that if α is the line segment $(-L/2, 0)$ to $(L/2, 0)$,

$$\iint I(p, \theta)\, \mathrm{d}p\, \mathrm{d}\theta = 2L.$$

  by doing the integral explicitly.

To write a corresponding integralgeometric formula for the length of a plane curve in terms of intersections with *circles*, we define the circle $C(r, x, y)$ to be the circle with radius r centered at (x, y). We will try to prove that

$$\mathrm{Length}(\alpha) = K \iint \mathrm{CI}(r, x, y)\, \mathrm{d}r\, \mathrm{d}x\, \mathrm{d}y.$$

where $\mathrm{CI}(r, x, y)$ is the number of intersections between α and the circle $C(r, x, y)$.

- Suppose $A_{\theta,\vec{v}}$ is a rigid motion of the plane which rotates the plane by θ and translates the plane by $\vec{v}$. Find the corresponding transformation $\tilde{A}$ of the space of circles and prove that the volume integral $\mathrm{d}r\, \mathrm{d}x\, \mathrm{d}y$ is unchanged by $\tilde{A}$.
- Let α be the line segment $(-L/2, 0)$ to $(L/2, 0)$, and attempt to compute the *improper* integral

$$\iint \mathrm{CI}(r, x, y)\, \mathrm{d}r\, \mathrm{d}x\, \mathrm{d}y.$$

  Is it well-defined? If so, is it equal to a constant multiple of L?
- Depending on your answer above, either complete the proof of the (conjectured) circle intersection formula above or discuss why it cannot be done.

2. A curve $\alpha(s)$ has $\alpha(0) = (0, 0, 0)$, $T(0) = (1, 0, 0)$, $N(0) = (0, 1, 0)$ and $B(0) = (0, 0, 1)$. The curve has curvature given by the function $\kappa(s) = s^2 \cos s + 3$ and torsion given by the function $\tau(s) = e^s \sin s$.
   (1) Find $\alpha(2.0)$.
   (2) Make a 3d model of the curve on the interval $[0, 2]$ and plot it from several views.

   This problem certainly requires *Mathematica* or other computer program to solve, along with a numerical method. I don't think that you can do the integrals symbolically.

3. We have constructed the Frenet frame of a curve $\alpha(s)$ using the procedure in notes 2, and defined the curvature and torsion accordingly. This question describes an alternate construction of the Frenet frame.

Definition 1. *A* general curve of type n *is an arclength-parametrized curve* $\alpha(s) \subset \mathbb{R}^n$ *where the* n *vectors*

$$\mathcal{B} = (\vec{\alpha}'(s), \vec{\alpha}''(s), \ldots, \vec{\alpha}^{(n)}(s))$$

are linearly independent (and hence a basis for $\mathbb{R}^n$*). We say that* Frenet's moving basis $\mathcal{T}$ *for a general curve of type* n *is the orthonormal basis*

$$\mathcal{T} = (\vec{t}_1(s), \ldots, \vec{t}_n(s))$$

for $\mathbb{R}^n$ *generated by applying Gram-Schmidt orthonormalization to the basis* $\mathcal{B}$*.*

(1) Prove that Frenet's moving basis $\mathcal{T} = (\vec{t}_1, \vec{t}_2, \vec{t}_3)$ for a regular curve of type 3 is the ordinary Frenet frame.

(2) Consider a regular curve of type 4. There exists a 4×4 matrix A of coefficients α_{ij} so that for each i,

$$\frac{\mathrm{d}}{\mathrm{d}s}\vec{t}_i = \sum \alpha_{ij}\vec{t}_j.$$

Prove that A is a skew-symmetric matrix with only 3 nonzero entries above the main diagonal. Where are they?

(3) Prove that two of the nonzero entries of A above the diagonal must be positive, while the third can have either sign. The first two entries are called the *curvatures* of the curve α while the third is called the *torsion*. Which entry in the matrix A is the torsion?