Math 4250/6250 Homework #1

This homework assignment covers DoCarmo 1.1 - 1.5. It accompanies Lectures 1 and 2 in the course notes. Please pick 5 of the following 9 problems. Remember that undergraduate students should average one challenge problem per assignment, while graduate students should average two challenge problems per assignment.

1. REGULAR PROBLEMS

1. (Do Carmo, 1-3, #2) A circular disk of radius 1 in the $xy$ plane rolls along the $x$ axis without slipping. The curve described by a point on the rim of the disk is called a cycloid.
   (1) Find a parametrization $\alpha(t)$ of the cycloid.
   (2) Compute the arclength of the portion of the cycloid corresponding to one complete rotation of the disk.

2. (Do Carmo, 1-3, #4) The curve $\alpha(t) = \left( \sin t, \cos t + \log \tan \frac{t}{2} \right)$ is called the tractrix. Show that
   (1) $\alpha$ is a differentiable parametrized curve, regular except at $t = \pi/2$.
   (2) The length of the portion of the tangent line to the tractrix between $\alpha(t)$ and the $y$-axis is always equal to 1.

3. (Based on Do Carmo, 1-5, #3) Given a curve $\alpha(s)$ parametrized by arclength, consider the curve $T(s)$ on the unit sphere. This is called the tangent indicatrix of $\alpha$. Prove that the speed of $T(s)$ is equal to the curvature of $\alpha$. The curve $N(s)$ is called the normal indicatrix of $\alpha(s)$. Prove that the speed of $N(s)$ is equal to the length of the vector $(\kappa(s), \tau(s)) \in \mathbb{R}^2$. The curve $B(s)$ is called the binormal indicatrix. Prove that the speed of $B(s)$ is $|\tau(s)|$.

2. CHALLENGE PROBLEMS

1. (Do Carmo 1-3, #8). Let $\alpha : I \rightarrow \mathbb{R}^3$ be a differentiable (that is, $C^\infty$) regular curve and let $[a, b]$ be a closed interval. For every partition $P = a = t_0 < t_1 < \cdots < t_n = b$ of $[a, b]$, let
   \[ \ell(P) = \sum |\alpha(t_{i+1}) - \alpha(t_i)|. \]
   Let the mesh of the partition be $|P| = \max t_{i+1} - t_i$. Prove that for any $\epsilon > 0$, there exists $\delta > 0$ so that if $|P| < \delta$ then
   \[ \left| \int_a^b |\alpha'(t)| \, dt - \ell(P) \right| < \epsilon. \]
   That is, the lengths of polygons inscribed in the curve converge to the length of the curve.

2. (Based on Do Carmo 1-3, #10). Let $\alpha : I \rightarrow \mathbb{R}^3$ be a a differentiable parametrized curve. Suppose $[a, b] \in I$ and $\alpha(a) = p$ while $\alpha(b) = q$.
   (1) Show that for any constant vector $v$ with $|v| = 1$,
   \[ \langle q - p, v \rangle \int_a^b \langle \alpha'(t), v \rangle \, dt \leq \int_a^b |\alpha'(t)| \, dt. \]
(2) Let
\[ v = \frac{q - p}{|q - p|} \]
and show that
\[ |\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)| \, dt. \]
That is, the curve of shortest length joining two points is the straight line!

3. Using the setup of the last problem, suppose that \( p \) lies in the plane \( z = 0 \) (that is, \( p = (p_1, p_2, 0) \)) and \( q \) lies in the plane \( z = 1 \) (that is, \( q = (q_1, q_2, 1) \)). Prove that the shortest curve joining any such \( p \) and \( q \) is the straight line joining \( p = (x, y, 0) \) to \( q = (x, y, 1) \).

4. Prove that a nonplanar curve with curvature \( \kappa(s) \) and torsion \( \tau(s) \) lies entirely on a sphere if and only if
\[ \frac{\tau(s)}{\kappa(s)} = \frac{d}{ds \left( \frac{\kappa'(s)}{\tau(s)\kappa^2(s)} \right)} \]

5. If \( \gamma(s) \) is an arclength-parametrized curve with nonzero curvature, find a vector \( \omega(s) \), expressed as a linear combination of \( T, N, \) and \( B \) so that
\[ T'(s) = \omega(s) \times T(s) \]
\[ N'(s) = \omega(s) \times N(s) \]
\[ B'(s) = \omega(s) \times B(s) \]
This vector is called the **Darboux vector**. Find a formula for the length of the Darboux vector in terms of the curvature \( \kappa(s) \) and torsion \( \tau(s) \) of the curve.