Math 4250/6250 Homework #1

This homework assignment covers DoCarmo 1.1 - 1.5. It accompanies Lectures 1 and 2 in the course notes. Please pick 5 of the following 9 problems. Remember that undergraduate students should average **one** challenge problem per assignment, while graduate students should average **two** challenge problems per assignment.

1. Regular Problems

1. (Do Carmo, 1-3, #2) A circular disk of radius 1 in the xy plane rolls along the x axis without slipping. The curve described by a point on the rim of the disk is called a *cycloid*.
   (1) Find a parametrization $\alpha(t)$ of the cycloid.
   (2) Compute the arclength of the portion of the cycloid corresponding to one complete rotation of the disk.

2. (Do Carmo, 1-3, #4) The curve
$$\alpha(t) = \left(\sin t, \cos t + \log \tan \frac{t}{2}\right).$$
is called the tractrix. Show that
   (1) α is a differentiable parametrized curve, regular except at $t = \pi/2$.
   (2) The length of the portion of the tangent line to the tractrix between $\alpha(t)$ and the y-axis is always equal to 1.

3. (Based on Do Carmo, 1-5, #3) Given a curve $\alpha(s)$ parametrized by arclength, consider the curve $T(s)$ on the unit sphere. This is called the *tangent indicatrix* of α. Prove that the speed of $T(s)$ is equal to the curvature of α. The curve $N(s)$ is called the *normal indicatrix* of $\alpha(s)$. Prove that the speed of $N(s)$ is equal to the length of the vector $(\kappa(s), \tau(s)) \in \mathbf{R}^2$. The curve $B(s)$ is called the *binormal indicatrix*. Prove that the speed of $B(s)$ is $|\tau(s)|$.

2. Challenge Problems

1. (Do Carmo 1-3, #8). Let $\alpha\colon I \to \mathbf{R}^3$ be a differentiable (that is, C^∞) regular curve and let $[a, b]$ be a closed interval. For every partition $P = a = t_0 < t_1 < \cdots < t_n = b$ of $[a, b]$, let
$$\ell(P) = \sum |\alpha(t_{i+1}) - \alpha(t_i)|.$$
Let the *mesh* of the partition be $|P| = \max t_{i+1} - t_i$. Prove that for any $\epsilon > 0$, there exists $\delta > 0$ so that if $|P| < \delta$ then
$$\left| \int_a^b |\alpha'(t)|\, dt - \ell(P) \right| < \epsilon.$$
That is, the lengths of polygons inscribed in the curve converge to the length of the curve.

2. (Based on Do Carmo 1-3, #10). Let $\alpha\colon I \to \mathbf{R}^3$ be a a differentiable parametrized curve. Suppose $[a, b] \in I$ and $\alpha(a) = p$ while $\alpha(b) = q$.
   (1) Show that for any constant vector v with $|v| = 1$,
$$\langle q - p, v\rangle \int_a^b \langle \alpha'(t), v\rangle \; dt \leq \int_a^b |\alpha'(t)|\, dt.$$

(2) Let
$$v = \frac{q-p}{|q-p|}$$
and show that
$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)|\, dt.$$
That is, the curve of shortest length joining two points is the straight line!

3. Using the setup of the last problem, suppose that p lies in the plane $z = 0$ (that is, $p = (p_1, p_2, 0)$) and q lies in the plane $z = 1$ (that is, $q = (q_1, q_2, 1)$). Prove that the shortest curve joining any such p and q is the straight line joining $p = (x, y, 0)$ to $q = (x, y, 1)$.

4. Prove that a nonplanar curve with curvature $\kappa(s)$ and torsion $\tau(s)$ lies entirely on a sphere if and only if
$$\frac{\tau(s)}{\kappa(s)} = \frac{d}{ds}\left(\frac{\kappa'(s)}{\tau(s)\kappa^2(s)}\right)$$

5. If $\gamma(s)$ is an arclength-parametrized curve with nonzero curvature, find a vector $\omega(s)$, expressed as a linear combination of T, N, and B so that
$$T'(s) = \omega(s) \times T(s)$$
$$N'(s) = \omega(s) \times N(s)$$
$$B'(s) = \omega(s) \times B(s)$$
This vector is called the *Darboux vector*. Find a formula for the length of the Darboux vector in terms of the curvature $\kappa(s)$ and torsion $\tau(s)$ of the curve.