Math 4250/6250 Homework #3

This homework assignment covers Section 1.3 in Shifrin.

Shifrin, Exercises 1.3: 1, 2, 4, 6, 7, 8.

In addition, answer the following question:

If $\gamma(s)$ is an arclength-parametrized curve with nonzero curvature, find a vector $\omega(s)$, expressed as a linear combination of T, N, and B so that

$$T'(s) = \omega(s) \times T(s)$$
$$N'(s) = \omega(s) \times N(s)$$
$$B'(s) = \omega(s) \times B(s)$$

This vector is called the *Darboux vector*. Find a formula for the length of the Darboux vector in terms of the curvature $\kappa(s)$ and torsion $\tau(s)$ of the curve.

Challenge Problem for Graduate Students

There's a natural intuition that making a curve of fixed length "three dimensional" involves crumpling the curve and reducing distances between corresponding points (at least on average). This intuition is captured by *Sallee's Stretching Theorem*:

Theorem 1 (Sallee)**.** *If γ is a closed space curve, there is a corresponding closed plane curve γ^* with the same length so that*

$$|\gamma(s) - \gamma(t)| \leq |\gamma^*(s) - \gamma^*(t)|$$

for every pair s, t.

In this question, you'll prove the theorem. Suppose you have a polygon V in space with vertices $v_1, \dots, v_n$. Connecting each pair of vertices $v_i v_{i+1}$ to a point p gives a collection of triangles called the "cone of V to p".

(1) The sum of the angles of the vertices of triangles at p is called the intrinsic cone angle $\theta(p)$. Prove that if p is inside the convex hull of V then $\theta(p) \geq 2\pi$.
(2) Now prove that for some p_0, we have $\theta(p_0) = 2\pi$. Hint: Consider the cone angle when p is very far from V.
(3) Now construct a planar polygon V^* by laying the triangles of the cone at p_0 into the plane around the origin. Show that V^* is closed and has the same length as V.
(4) Last, prove the theorem for V and V^*: if both polygons are parametrized by arclength starting at the same vertex v_1, then $|V(s) - V(t)| \geq |V^*(s) - V^*(t)|$ for any s, t. Extend the theorem to smooth curves by approximation.