Math 4250/6250 Homework #3

This homework assignment covers our notes on integral geometry (5) and on rotation index (6). Please pick 3 of the following problems. Remember that undergraduate students should average **one** challenge problem per assignment, while graduate students should average **two** challenge problems per assignment.

1. Regular Problems

1. Here are two related problems about length and curvature:
   - Suppose that $\alpha(s)$ is a simple closed plane curve with curvature $0 < \kappa(s) < 1/R$ (that is, curvature *less* than the curvature of a circle of radius R. Prove that
   $$\text{Length}(\alpha) \geq 2\pi R.$$
   - Suppose that $\alpha(s)$ is a curve of rotation index N with curvature $0 < \kappa(s) < 1/R$. Prove that
   $$\text{Length}(\alpha) \geq N 2\pi R.$$
2. Consider a unit circle C in the plane. Let S be the set of straight lines which intersect C and let S' be the set of straight lines which cut C in a chord of length $> \sqrt{3}$ (that is, a chord longer than the side of an equilateral triangle inscribed in C). Remember that we can parametrize lines in the plane by two coordinates: θ and p. Now for any set L of lines in the plane, we can define the "measure" (we know this as the *area*) of the set of lines to be the integral
   $$M(L) = \int_{\ell(\theta,p)\in L} 1 \, \mathrm{d}p \, \mathrm{d}\theta.$$
   For our sets S and S' above, prove that $M(S')/M(S) = 1/2$. This shows that in a precise sense, half of the lines intersecting the circle make a chord larger than $\sqrt{s}$.
3. The curve $\alpha(t) = ((2a\cos t + b)\cos t, (2a\cos t + b)\sin t)$ with $t \in [0, 2\pi)$ is called a limacon. Compute the rotation index of this curve.
4. Suppose that $\alpha(s)$ is a closed convex plane curve. Define the *parallel curve* at distance r to be
   $$\beta(s) = \alpha(s) - r\vec{n}(s)$$
   where $\vec{n}(s)$ is the (unit) normal vector to α. If $\kappa_\beta(s)$ is the curvature of $\beta(s)$ and $\kappa_\alpha(s)$ is the curvature of $\alpha(s)$, prove that
   - $\text{Length}(\beta) = \text{Length}(\alpha) + 2\pi r$.
   - $\text{Area}(\beta) = \text{Area}(\alpha) + r\,\text{Length}(\alpha) + \pi r^2$.
   - $\kappa_\beta(s) = \kappa_\alpha(s)/(1 + r\kappa_\alpha(s))$.

2. CHALLENGE PROBLEMS

1. (Curves of Finite Total Curvature). Suppose $a(s) : S^1 \to \mathbf{R}^2$ is a smooth, regular closed curve of length ℓ parametrized by arclength. Let a subdivision $\mathcal{S}_n$ of a be a collection of parameter values $x_0 = 0 < x_1 < \cdots < x_n < \ell$. Let the mesh size $\mathrm{Mesh}(\mathcal{S}_n)$ of the subdivision $\mathcal{S}_n$ be the maximum of $x_i - x_{i-1}$. The exterior angle or turning angle θ_i of the subdivision at i is the angle formed by $a(x_{i-1})a(x_i)$ and $a(x_i)a(x_{i+1})$.

   If $\kappa(s)$ is the curvature of $a(s)$, then the total curvature of a is given by
   $$K = \int \kappa(s)\,\mathrm{d}s.$$
   Prove that
   $$K = \lim_{\mathrm{Mesh}(\mathcal{S}_n) \to 0} \sum_{i=0}^{n} \theta_i.$$

2. Prove Istvan Fary's integralgeometric formula for curvature. If $a(s)$ is a space curve and $a_v(s)$ is the projection of $a(s)$ to the plane through the origin normal to v, let $\kappa(s)$ denote the curvature of $a(s)$ and $\kappa_v(s)$ denote the curvature of $a_v(s)$. And let K_v be the total curvature of $a_v(s)$ and K be the total curvature of so that
   $$K_v = \int \kappa_v(s)\,\mathrm{d}s \quad \text{and} \quad K = \int \kappa(s)\,\mathrm{d}s.$$
   Now show that
   $$K = C \int_{S^2} K_v \,\mathrm{dArea}$$
   where C is a constant, and v is integrated over S^2.

   Hint: Use problem 1 to reduce the problem to the case where a is a polygon. Show first that the total curvature of such a curve formed by two line segments w_1 and w_2 is the angle between the tangents to w_1 and w_2.

   Hint 2: Suppose that $\theta = \angle x_1 x_2 x_3$ is the angle between $x_2 x_1$ and $x_2 x_3$, and that θ_v is the angle between the projection of x_1, x_2, and x_3 into the plane normal to v. To complete hint 1, you must show that
   $$\theta = C\,\mathrm{Avg}(\theta) = C \int_{v \in S^2} \theta_v \,\mathrm{dArea}.$$
   Instead of doing the integral on the right directly, try to prove that the function $\mathrm{Avg}(\theta)$ is a linear function of θ. Can you compute $\mathrm{Avg}(0)$ and $\mathrm{Avg}(\pi)$?

3. Prove Milnor's integralgeometric formula for curvature. If $a(s)$ is a space curve with curvature $\kappa(s)$, let $a_v(s)$ be the projection of $a(s)$ to a straight line. This is a nonregular curve with total curvature $K_v = \pi \cdot$ (the # of times the curve changes direction). Prove that
   $$\int \kappa(s)\,\mathrm{ds} = K \int_{v \in S^2} k_v \,\mathrm{dArea}.$$

4. In exercise #4 of the regular problems we showed that one could define the **parallel curve** to a **smooth** convex curve $\alpha(t)$ by constructing the curve
   $$\beta(t) = \alpha(t) - rN(t)$$
   and that we could prove $\mathrm{Length}(\beta) = \mathrm{Length}(\alpha) + 2\pi r$ using differential geometry.

Suppose now that the curve $\alpha(t)$ is convex, but not smooth (like a square) and redefine the parallel to α to be the outer boundary curve of the set of points within distance r of the curve α. Prove that (as before)

$$\text{Length}(\beta) = \text{Length}(\alpha) + 2\pi r,$$

this time using integral geometry.