Math 4250/6250 Homework #4

This homework assignment covers our notes on regular surfaces (7-8), tangent planes (9) and the first fundamental form (10). Please pick 5 of the following problems. Remember that undergraduate students should average **one** challenge problem per assignment, while graduate students should average **two** challenge problems per assignment. **Everyone should complete the required problems.**

1. Regular Problems

1. Consider the sphere $x^2+y^2+(z-1)^2 = 1$ centered at $(0,0,1)$ with radius 1. We can construct a (very important) map π from this sphere to the x-y plane by defining $\pi(p)$ to be the intersection of the line through $p \in S^2$ and $(0,0,2)$ with the x-y plane. We note that π is not defined at the north pole $(0,0,2)$.
   (1) Show that the inverse map $\pi^{-1}\colon\ \mathbb{R}^2 \to S^2 - (0,0,2)$ is defined by
   $$\pi^{-1}(u,v) = \left(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{2(u^2+v^2)}{u^2+v^2+4}\right)$$
   (2) Show that the inverse map π^{-1} provides a regular parametrization of $S^2 - (0,0,2)$.
   (3) Find the first fundamental form I_p of this parametrization as a matrix.
2. Consider the hyperboloid of revolution S defined by $x^2+y^2-z^2 = 1$. This surface intersects the x-y plane in the circle $x^2+y^2 = 1$. Prove that the tangent planes of S along this circle are all parallel to the z axis.
3. Suppose that S is parametrized by a map $X : \mathbb{R}^2 \to S$ in the form
   $$X(u,v) = \alpha_1(u) + \alpha_2(v)$$
   where α_1 and α_2 are regular curves. For instance, if $\alpha_1(u) = (\cos u, \sin u, 0)$ and $\alpha_2(v) = (0,0,v)$, the surface is the cylinder. Show that the tangent planes along the curve
   $$\beta(s) = X(s, v_0)$$
   are parallel to a line. What is the line?
4. Let $\alpha : [0,1] \to \mathbb{R}^3$ be a regular parametrized curve with unit tangent vector $T(s)$. We say that $N_1, N_2 : [0,1] \to \mathbb{R}^3$ form a *framing* for α if $(T(s), N_1(s), N_2(s))$ is an orthonormal basis for $\mathbb{R}^3$ for every s. The *tube around* α *of radius* r is the surface parametrized by
   $$X(u,v) = \alpha(u) + \cos v N_1(u) + \sin v N_2(u).$$
   Find the normal vector $N(u,v)$ of the tube.
5. **(Required Problem)** Find the first fundamental form for the surfaces below (a, b, and c are constants):
   (1) The ellipsoid. $X(u,v) = (a \sin u \cos v, b \sin u \sin v, c \cos u)$.
   (2) The elliptic paraboloid. $X(u,v) = (au \cos v, bu \sin v, u^2)$.
   (3) The hyperboloid paraboloid. $X(u,v) = (au \cosh v, bu \sinh v, u^2)$.
   (4) The hyperboloid of two sheets. $X(u,v) = (a \sinh u \cos v, b \sinh u \sin v, c \cosh u)$.

2. Challenge Problems

1. Suppose we have an arclength-parametrized curve $C(s) = (x(s), 0, z(s))$ in the x-z plane which does not meet the z axis. We can form the corresponding *surface of revolution S generated by C* by rotating C around the z axis. This surface is parametrized by
$$X(u, v) = (x(v)\cos u, x(v)\sin u, z(v)).$$
   (1) (*Pappus' Theorem*) Show that the area of S is given by
$$A(S) = 2\pi \int_0^\ell x(s)\, ds.$$
   where ℓ is the length of C.
   (2) Suppose C is the circle of radius r_1 in the x-z plane centered at $(r_2, 0, 0)$ (and $r_2 > r_1$). Use the first part of this problem to compute the area of the torus of revolution generated by C.
2. Generalize the last problem to show that the area of a tube of radius r around a curve α (cf. Problem 4 above) is $2\pi r$ times the length of α.
3. **Required Problem for Graduate Students: Extrinsic and Intrinsic Gradient.** We might remember that a function $f(\vec{x})$ on $\mathbf{R}^n$ has a directional derivative in any direction $\vec{v}$ given by the limit
$$Df(\vec{v}) = \lim_{h\to 0} \frac{f(\vec{x} + h\vec{v}) - f(\vec{x})}{h}.$$
We learn in multivariable calculus that this directional derivative is a linear function of v, and so that Df is a linear functional on the space of direction vectors v. In fact, there is a special vector $\nabla f = (\partial f/\partial x_1, \dots, \partial f/\partial x_n)$ so that
$$Df(v) = \langle v, \nabla f\rangle .$$
If the function $f(\vec{x})$ is defined on a curved surface S, we still want to be able to understand what it means to differentiate the function. In fact, the directional derivative at p is a linear function of directions in the tangent plane T_pS. This linear functional is now written as
$$Df(\vec{v}) = \langle \vec{v}, \nabla f\rangle_{I_p} .$$
where ∇f is a 2-vector in T_pS given in X_u, X_v coordinates. This vector is called the *intrinsic gradient* of f. But what is the formula for ∇f? It is important to know it. But it is clearly not as simple as it used to be for functions defined on R^n. Finding a formula for ∇f will require us to understand the first fundamental form in some detail, using the theory we've developed. This leads us to an answer to our question "What is differential geometry for?":

Differential geometry tells you how to do calculus on a curved surface.

Let's begin. The *extrinsic gradient* of a differentiable function $f\colon S \to \mathbb{R}$ is a differentiable map $\nabla f\colon S \to \mathbb{R}^3$ which assigns to each point of S a vector $\nabla f(p)$ so that
$$Df_p(\vec{v}) = \langle \nabla f, \vec{v}\rangle_{\mathbb{R}^3} .$$
It's worth noting that this is the $\mathbb{R}^3$ inner product because we wrote the gradient as a 3-vector in space. Because we're referring to the "external" space $\mathbb{R}^3$, we call this gradient "extrinsic". Show that

(1) If E, F, G are the coefficients of the first fundamental form on S, then *as a vector in* $\mathbb{R}^3$ the extrinsic gradient is

$$\nabla f = \frac{f_u G - f_v F}{EG - F^2} \vec{X}_u + \frac{f_v E - f_u F}{EG - F^2} \vec{X}_v.$$

and hence the *intrinsic* gradient of f at p is

$$\nabla f = \left(\frac{f_u G - f_v F}{EG - F^2}, \frac{f_v E - f_u F}{EG - F^2} \right).$$

(2) Suppose S is the x-y plane with the parametrization $X(u, v) = (u, v, 0)$. Compute the coefficients of the first fundamental form and use the formula above to show that the intrinsic gradient $\nabla f = (f_u, f_v)$.

(3) Fix a p in S and consider the unit circle $|\vec{v}| = 1$ in $T_p(S)$. Prove that on this circle $Df_p(\vec{v})$ is maximized $\iff v = \nabla f / |\nabla f|$.

(4) Consider a *level curve* $C = \{p \in S : f(p) = c\}$ on S. Prove that ∇f is normal to C everywhere.