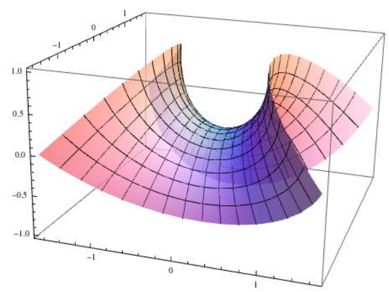
Math 4250/6250 Homework #6

This homework assignment covers our notes on the meaning of the second fundamental form (13), computations with the second fundamental form in local coordinates (14), and extracting geometric information from the second fundamental form (15). Please pick 4 of the following problems, **including problem 3 of the regular problems**. Remember that undergraduate students should average **one** challenge problem per assignment, while graduate students should average **two** challenge problems per assignment.

1. Regular Problems

1. Find the asymptotic curves and lines of curvature of the surface $X(u, v) = (u, v, uv)$.
2. Consider Enneper's surface

$$X(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right).$$



Show that

(1) The first fundamental form is given by

$$E = G = (1 + u^2 + v^2)^2, \quad F = 0.$$

(2) The second fundamental form is given by

$$e = 2, \quad g = -2, \quad f = 0.$$

(3) The principal curvatures are

$$k_1 = \frac{2}{(1 + u^2 + v^2)^2}, \quad k_2 = -\frac{2}{(1 + u^2 + v^2)^2}.$$

(4) Compute the Gauss and Mean curvature of this surface.

3. Suppose that we are given a surface of revolution

$$X(u, v) = (\phi(v) \cos u, \phi(v) \sin u, \psi(v))$$

so that the profile curve $\alpha(v) = (\phi(v), 0, \psi(v))$ is parametrized by arclength (that is $(\phi')^2 + (\psi')^2 = 1$). We want to solve for $\psi(v)$ and $\phi(v)$ so that the surface has constant Gauss curvature K.

(1) Show that $\phi(v)$ and $\psi(v)$ satisfy the equations

$$\phi''(v) + K\phi(v) = 0, \qquad \psi(v) = \int \sqrt{1 - (\phi')^2}\, \mathrm{d}v.$$

(2) Suppose that $K = 1$. Show that if we assume that $\psi'(0) = 0$ then the solutions of the equations above are

$$\phi(v) = C\cos v, \qquad \psi(v) = \int_0^v \sqrt{1 - C^2 \sin^2 t}\, \mathrm{d}t.$$

Here $C = \phi(0)$ is some constant. Notice that $\psi(v)$ is *not* defined for all v– we can only integrate where $1 - C^2 \sin^2 t$ is positive. First, find the domain of $\psi(v)$ (depending on C). Sketch the profile curve $\alpha(s) = (\phi(v), 0, \psi(v))$ for $C < 1$, $C = 1$, and $C > 1$ and the resulting surface of revolution (you can use Wolfram Alpha to make plots if you don't have Mathematica). Next, show that only the $C = 1$ surface can be reflected over the xy plane to make a compact regular surface.

(3) Now consider the case $K = -1$. Show that either

$$\phi(v) = C\cosh v, \qquad \psi(v) = \int_0^v \sqrt{1 - C^2 \sinh^2 t}\, \mathrm{d}t.$$

or

$$\phi(v) = C\sinh v, \qquad \psi(v) = \int_0^v \sqrt{1 - C^2 \cosh^2 t}\, \mathrm{d}t.$$

or

$$\phi(v) = e^v, \qquad \psi(v) = \int_0^v \sqrt{1 - e^{2t}}\, \mathrm{d}t.$$

In each case, determine the range of v values for which $\psi(v)$ makes sense and sketch the resulting surface. Again, you can use Wolfram Alpha to make plots if you want to.

(4) Last, consider the case $K = 0$. Prove that the only solutions are the cylinder, the cone, and the plane.

2. Challenge Problems

1. Let $h : S \to \mathbb{R}$ be a differentiable function on a surface S and let $p \in S$ be a critical point of h. Let $\vec{w} \in T_pS$ and let $\alpha : (-\epsilon, \epsilon) \to S$ be a curve in S with $\alpha(0) = p$ and $\alpha'(0) = \vec{w}$. We define

$$H_p h(\vec{w}) = \frac{d^2}{dt^2} h(\alpha(t))\Big|_{t=0}$$

Then

(1) Let x be a parametrization of S with $X(0,0) = p$. Show that

$$H_p h(u'\vec{x_u} + v'\vec{x_v}) = h_{uu}(p)(u')^2 + 2h_{uv}u'v' + h_{vv}(p)(v')^2.$$

Prove that $H_p h : T_pS \to \mathbb{R}$ is a quadratic form on T_pS which does not depend on our choice of α. (This will depend on the fact that p is a critical point of h.) This quadratic form is called the *Hessian* of h at p.

(2) Let $h : S \to \mathbb{R}$ be the height function of S relative to T_pS. That is, if $\vec{n}$ is the normal vector to S at p,

$$h(\vec{q}) = \langle \vec{q} - \vec{p}, \vec{n} \rangle .$$

Check that p is a critical point of h with $h(\vec{p}) = 0$, and so that $H_p h$ is well defined. Now prove that if $\vec{w} \in T_pS$ and $|\vec{w}| = 1$ then

$$H_p h(\vec{w}) = \text{the normal curvature of } S \text{ at } p \text{ in the direction } \vec{w}.$$

(3) Conclude that the Hessian of the height function relative to T_pS is the second fundamental form of S at p.

2. A critical point $p \in S$ of a differentiable function $h : S \to \mathbb{R}$ is *nondegenerate* if the matrix A_p associated to the quadratic form $H_p h$ is nonsingular (that is, each eigenvalue of the matrix is positive or negative, and there are no zero eigenvalues). A differentiable function h on S for which every critical point is nondegenerate is called a *Morse function*. The point of this exercise is to show that Morse functions are "common" on surfaces in the sense that the distance function from S to almost every point $\vec{r} \in \mathbb{R}^3$ is a Morse function. Given $\vec{r}$, we let

$$h_{\vec{r}}(\vec{q}) = \sqrt{\langle \vec{q} - \vec{r}, \vec{q} - \vec{r} \rangle}.$$

We now show that this is a Morse function for almost every $\vec{r}$.

(1) Show that $\vec{p} \in S$ is a critical point for $h_{\vec{r}}$ if and only if the line $\vec{p}\vec{r}$ is normal to S at $\vec{p}$.

(2) Suppose that $\vec{p}$ is a critical point for $h_{\vec{r}}$. Given a direction $\vec{w} \in T_pS$ with $|\vec{w}| = 1$, prove that

$$H_p h_{\vec{r}}(\vec{w}) = \frac{1}{h_{\vec{r}}(p)} - \mathrm{II}_p(\vec{w}).$$

where II_p is the second fundamental form of S.

(3) Prove that if $\vec{e}_1$ and $\vec{e}_2$ are the principal directions for S at p then $\vec{e}_1$ and $\vec{e}_2$ are eigenvectors for A_p.

(4) Suppose k_1 and k_2 are the principal curvatures of S at p. Prove that p is a degenerate critical point of $h_{\vec{r}}$ if and only if either

$$h_{\vec{r}}(p) = \frac{1}{k_1} \quad \text{or} \quad h_{\vec{r}}(p) = \frac{1}{k_2}$$

(5) Now show the set of $\vec{r} \in \mathbb{R}$ for which $h_{\vec{r}}$ is a Morse function is open and dense in $\mathbb{R}^3$.

3. A surface $S \subset \mathbb{R}^3$ is *locally convex* at a point $\vec{p} \in S$ if there exists a neighborhood V of $\vec{p}$ in S so that V is contained in one of the closed half-spaces determined by $T_{\vec{p}}S$. If V has only $\vec{p}$ in common with $T_{\vec{p}}S$ then we say that S is *strictly locally convex* at $\vec{p}$.

(1) Prove that if the principal curvatures of S are nonzero and have the same sign at $\vec{p}$ (and hence the Gauss curvature $K > 0$), then S is strictly locally convex at $\vec{p}$.

(2) Prove that if S is locally convex at $\vec{p}$ then the principal curvatures of S do not have different signs (so $K \geq 0$).

(3) In fact, $K \geq 0$ does **not** imply local convexity! To see this, consider the surface

$$X(u, v) = (u, v, u^3(1 + v^2)).$$

Prove that the Gauss curvature of X is nonnegative on $U = \{(u, v) \in \mathbb{R}^2 | v^2 < 1/2\}$, but that S is not locally convex at $(0, 0)$.