1. **Regular Problems**

1. Show that if $F = 0$ then we can compute Gauss curvature $K$ with the simple formula

\[ K = -\frac{1}{2\sqrt{EG}} \left[ \frac{d}{dv} \left( \frac{E_v}{\sqrt{EG}} \right) + \frac{d}{du} \left( \frac{G_u}{\sqrt{EG}} \right) \right]. \]

2. Show that if $X$ is an isothermal parametrization, that is, if $E = G = \lambda(u, v)$ and $F = 0$, then we can compute $K$ with the even simpler formula

\[ K = -\frac{1}{2\lambda} \Delta \log \lambda \]

where the Laplacian $\Delta$ is defined by

\[ \Delta \lambda = \lambda_{uu} + \lambda_{vv}. \]

Now use this to show that if $E = G = (u^2 + v^2 + c)^{-2}$ for some constant $c$ and $F = 0$ then $K = 4c$.

3. Let $S$ be an oriented regular surface and let $\alpha : I \rightarrow S$ be an arclength-parametrized curve. The fact that $\alpha$ is contained in the surface means that $\alpha'(s) = T(s)$ is contained in the tangent plane $T_{\alpha(s)}S$ and hence is perpendicular to the surface normal $N_S(s)$. We can then define the **Darboux frame** on $\alpha$ to be the triple of vectors

\[ \text{Darboux frame} = (T(s), V(s) = N_S(s) \times T(s), N_S(s)). \]

This is a frame like the Frenet frame whose derivatives tell us about the local geometry of the curve as it lies in the surface $S$. Show first that in analogy to the Frenet equations we have functions $a(s), b(s),$ and $c(s)$ so that

\[
\begin{align*}
T' &= 0 + a(s)V(s) + N_S(s) \\
V' &= -a(s)T + 0 + c(s)N_S(s) \\
N_S' &= -b(s)T - c(s)V(s) + 0
\end{align*}
\]

We now need to interpret these coefficients geometrically. Show that

1. $c(s) = -\langle N'_S, V \rangle$. Hence $\alpha$ is a line of curvature $\iff c(s) = 0$. The function $-c(s)$ is called the **geodesic torsion** of $\alpha$.
2. $b(s)$ is the normal curvature $\kappa_n$ of $\alpha$.
3. $a(s)$ is the geodesic curvature $\kappa_g$ of $\alpha$.

4. Let $S$ be the hyperboloid of revolution

\[ X(u, v) = (\cosh v \cos u, \cosh v \sin u, \sinh v) \]
which is also given as an implicit surface by the equation \( x^2 + y^2 - z^2 = 1 \). Suppose that \( \alpha(s) \) is a geodesic on \( S \) which makes angle \( \theta(s) \) with the \( X_u \) direction at the point \( \alpha(s) = X(u, v) \). If \( \cos \theta(s) = 1 / \cosh v \), show that the geodesic spirals asymptotically towards the parallel \( v = 0 \).