Math 4250/6250 Homework #7

This homework assignment covers our notes on geodesics (18) and on the Gauss-Bonnet theorem (19). Please choose two problems, **including** #1.

1. Regular Problems

1. Show that if $F = 0$ then we can compute Gauss curvature K with the simple formula

$$K = -\frac{1}{2\sqrt{EG}}\left[\frac{d}{dv}\left(\frac{E_v}{\sqrt{EG}}\right) + \frac{d}{du}\left(\frac{G_u}{\sqrt{EG}}\right)\right].$$

2. Show that if X is an *isothermal parametrization*, that is, if $E = G = \lambda(u, v)$ and $F = 0$, then we can compute K with the even simpler formula

$$K = -\frac{1}{2\lambda}\Delta(\log\lambda)$$

where the *Laplacian* Δ is defined by

$$\Delta\lambda = \lambda_{uu} + \lambda_{vv}.$$

Now use this to show that if $E = G = (u^2 + v^2 + c)^{-2}$ for some constant c and $F = 0$ then $K = 4c$.

3. Let S be an oriented regular surface and let $\alpha : I \to S$ be an arclength-parametrized curve. The fact that α is contained in the surface means that $\alpha'(s) = T(s)$ is contained in the tangent plane $T_{\alpha(s)}S$ and hence is perpendicular to the surface normal $N_S(s)$. We can then define the *Darboux frame* on α to be the triple of vectors

$$\text{Darboux frame} = (T(s), V(s) = N_S(s) \times T(s), N_S(s)).$$

This is a frame like the Frenet frame whose derivatives tell us about the local geometry of the curve as it lies in the surface S. Show first that in analogy to the Frenet equations we have functions $a(s)$, $b(s)$, and $c(s)$ so that

$$\begin{aligned} T' &= 0 + a(s)V(s) + N_S(s) \\ V' &= -a(s)T + 0 + c(s)N_S(s) \\ N_S' &= -b(s)T - c(s)V(s) + 0 \end{aligned}$$

We now need to interpret these coefficients geometrically. Show that

(1) $c(s) = -\langle N_S', V\rangle$. Hence α is a line of curvature $\iff$ $c(s) = 0$. The function $-c(s)$ is called the *geodesic torsion* of α.
(2) $b(s)$ is the normal curvature κ_n of α.
(3) $a(s)$ is the geodesic curvature κ_g of α.

4. Let S be the hyperboloid of revolution

$$X(u, v) = (\cosh v \cos u, \cosh v \sin u, \sinh v)$$

which is also given as an implicit surface by the equation $x^2 + y^2 - z^2 = 1$. Suppose that $\alpha(s)$ is a geodesic on S which makes angle $\theta(s)$ with the X_u direction at the point $\alpha(s) = X(u, v)$. If $\cos\theta(s) = 1/\cosh v$, show that the geodesic spirals asymptotically towards the parallel $v = 0$.