1. Let \( A \) be an \( m \times n \) matrix with \( m \geq n \) and suppose \( A \) has full rank. Show that the equation
\[
\begin{pmatrix}
I & A \\
A^T & 0 \\
\end{pmatrix}
\begin{pmatrix}
r \\
x \\
\end{pmatrix} = \begin{pmatrix}
b \\
0 \\
\end{pmatrix}
\]
has a solution where \( x \) minimizes \( \|Ax - b\|_2 \).

2. Assuming as above that \( A \) is an \( m \times n \) matrix with \( m \geq n \) with full rank, what is the condition number of
\[
\begin{pmatrix}
I & A \\
A^T & 0 \\
\end{pmatrix}
\]
in terms of the singular values of \( A \)? (Hint: Use the SVD of \( A \).)

3. Assuming as above that \( A \) is an \( m \times n \) matrix with \( m \geq n \) with full rank, find an explicit expression for the inverse of
\[
\begin{pmatrix}
I & A \\
A^T & 0 \\
\end{pmatrix}
\]
as a block \( 2 \times 2 \) matrix. (Hint: Use \( 2 \times 2 \) block Gaussian elimination.)

4. (Bonus) Show how to use the QR decomposition of \( A \) to implement an iterative refinement algorithm to improve the accuracy of the solution for \( x \) in
\[
\begin{pmatrix}
I & A \\
A^T & 0 \\
\end{pmatrix}
\begin{pmatrix}
r \\
x \\
\end{pmatrix} = \begin{pmatrix}
b \\
0 \\
\end{pmatrix}
\]

5. Suppose that \( A \) is an \( m \times n \) matrix with \( \text{SVD} \, A = U\Sigma V^T \). Compute the SVDs of the following matrices in terms of \( U \), \( \Sigma \), and \( V \):
   (1) \( (A^T A)^{-1} \)
   (2) \( (A^T A)^{-1} A^T \)
   (3) \( A(A^T A)^{-1} \)
   (4) \( A(A^T A)^{-1} A^T \)

6. (Constrained Least Squares) Suppose we want to find \( x \) minimizing \( \|Ax - b\|_2 \) subject to the linear constraint \( Cx = d \). Suppose that \( A \) is \( m \times n \), \( C \) is \( p \times n \), and \( C \) has full rank. Suppose also that \( p \leq n \) (so that we can guarantee that \( Cx = d \) has a solution) and \( n \leq m + p \) (so that the system is not underdetermined).
   (1) Show that if
   \[
   \begin{pmatrix}
   A \\
   C \\
   \end{pmatrix}
   \]
has full column rank, then there is a unique solution.
   (2) (Bonus) Show how to compute the solution \( x \) using two QR decompositions, some matrix-vector multiplications, and some solutions of triangular system of linear equations. Hint: Look at the LAPACK routine \texttt{sgglse}. 