Math 4600 - Combinatorics.

Introduction. Counting. Principles and techniques more than theorems. Gas vs. diesel.

Products of sets. Given $\Omega_1, \ldots, \Omega_n$, the set $\Omega = \Omega_1 \times \cdots \times \Omega_n = \{ (\omega_1, \ldots, \omega_n) \mid \omega_i \in \Omega_i \}$. 

Fact. $\#\Omega = \#\Omega_1 \cdot \cdots \cdot \#\Omega_n$.

Example. Two students at UGA have the same 3 initials. Two students in the Redcoat marching band (pop 430) have the same birthday.

"Pigeonhole principle." If $f: \Omega_1 \to \Omega_2$ and $\#\Omega_2 < \#\Omega_1$, then $f$ is not 1-1.

We can improve this as follows.

Suppose we have $c$ students, and we consider the map $B: \Sigma_1, \Sigma_3 \to \Sigma_{x \cdot y \cdot z \cdot}$.
If we have $r$ students, their birthdays are an element of $\Omega \times \cdots \times \Omega = \Omega^r$ where $\Omega = \{\text{days in the year}\}$.

Q: What is the probability of the event $E = \{ (b_1, \ldots, b_r) \in \Omega^r \mid b_i = b_j \text{ for some } i \neq j \}$ assuming a uniform distribution in $\Omega^r$?

It's easier to compute $P(\neg E)$, with the following: If $(b_1, \ldots, b_r) \in \neg E$, then

\begin{align*}
b_1 &\in \Omega \\
b_2 &\in \Omega - \{b_1\} \\
b_3 &\in \Omega - \{b_1, b_2\} \\
&\vdots \\
b_r &\in \Omega - \{b_1, \ldots, b_{r-1}\}
\end{align*}

So

\[
\#(\neg E) = 365 \times 364 \times \cdots \times (365 - r + 1) = (365)_r \quad \text{"365 down } r"
\]
or a Pochhammer symbol.
Therefore
\[ P(\neg E) = \frac{\#(\neg E)}{\#(\Omega^c)} = \frac{(365)^r}{(365)^r} \]

Mathematica demo. Pochhammer \([365-r+1, r]/(365)^r\).

**Definition.** A permutation of \( \Omega \) is an element of \( \Omega^{\#\Omega} \) with no repeated entries.

\[ \text{Permutations}(\Omega) = \{ (\omega_1, \ldots, \omega_{\#\Omega}) \mid \omega_i \neq \omega_j \text{ if } i \neq j \} \]

Equivalently,
\[ \sigma \in \text{Permutations}(\Omega) \iff \sigma : \Omega \to \Omega \text{ is 1-1}. \]

Example. \( \Omega = \{a, b, c\} \), \( \sigma = (a \, b \, c) \).

If we order the elements of \( \Omega \), we can refer to them by numbers 1, \ldots, \#\Omega, and write
\[ \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \]

Proposition. \( \# \text{ Permutations}(\Omega) = n \cdot (n-1) \cdot \ldots \cdot 1 = n! \) if \( \#\Omega = n \).
Definition. A \( k \)-permutation of \( \Omega \) is an ordered subset of \( k \) elements from \( \Omega \) no repeats.

Lemma. If \( \# \Omega = n \), there are \( (n)_k \) \( k \)-permutations.

Factorials are hard to compute. Stirling's formula

\[
 n! \sim n^n e^{-n} \sqrt{2\pi n}
\]

Proposition. \( \lim_{n \to \infty} \frac{n^n e^{-n} \sqrt{2\pi n}}{n!} = 1 \).

Mathematica demo.

Definition. A permutation with \( o(\omega) = \omega \) has \( \omega \) as a fixed point. A permutation with no fixed points is called a derangement.

Mathematica demo. \( P(\text{derangement}) \to \frac{1}{e} \).

LengthPermutation Support @ RandomPermutation [n] == n.