Math 4600 - Combinatorics.

Introduction. Counting. Principles and techniques more than theorems. Gas vs. diesel.

Products of sets. Given $\Omega_1, \dots, \Omega_n$, the set $\Omega = \Omega_1 \times \dots \times \Omega_n = \{(\omega_1, \dots, \omega_n) \mid \omega_i \in \Omega_i\}$.

Fact. $\#\Omega = \#\Omega_1 \cdot \dots \cdot \#\Omega_n$.

Example. Two students at UGA have the same 3 initials. Two students in the Redcoat marching band (pop 430) have the same birthday.

"Pigeonhole principle." If $f: \Omega_1 \to \Omega_2$ and $\#\Omega_2 < \#\Omega_1$, then f is not 1-1.

We can improve this as follows.

~~Suppose we have r students, and~~

~~we consider the map~~ $\cancel{\text{Bday}: \{1, \dots, r\} \to Y \times \dots \times Y}$

If we have r students, their birthdays are an element of $\underbrace{\Omega \times \cdots \times \Omega}_{r} = \Omega^r$ where $\Omega = \{\text{days in the year}\}$.

Q: What is the probability of the event $E = \{(b_1, \ldots, b_r) \in \Omega^r \mid b_i = b_j \text{ for some } i \neq j\}$, assuming a uniform distribution in Ω^r?

It's easier to compute $P(\neg E)$, with the following: If $(b_1, \ldots, b_r) \in \neg E$, then

$$b_1 \in \Omega$$

$$b_2 \in \Omega - \{b_1\}$$

$$b_3 \in \Omega - \{b_1, b_2\}$$

$$\vdots$$

$$b_r \in \Omega - \{b_1, \ldots, b_{r-1}\}$$

So

$$\#(\neg E) = 365 \times 364 \times \cdots \times (365 - r + 1)$$

$$= (365)_r \quad \text{"365 down } r\text{"}$$

or a Pochhammer symbol.

Therefore

$$P(\neg E) = \frac{\#(\neg E)}{\#(\Omega^r)} = \frac{(365)_r}{(365)^r}$$

Mathematica demo. Pochhammer[365-r+1,r]/(365)^r

~~Per~~ Definition. A <u>permutation</u> of Ω is an element of $\Omega^{\#\Omega}$ with no repeated entries.

$$\text{Permutations}(\Omega) = \{(\omega_1, \dots, \omega_{\#\Omega}) \mid \omega_i \neq \omega_j \text{ if } i \neq j\}$$

Equivalently,

$$\sigma \in \text{Permutations}(\Omega) \Leftrightarrow \sigma : \Omega \to \Omega \text{ is 1-1.}$$

Example. $\Omega = \{a,b,c\}$ $\sigma = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$.

If we order the elements of Ω, we can refer to them by numbers $1, \dots, \#\Omega$, and write

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Proposition. $\#\text{Permutations}(\Omega) = n \cdot (n-1) \cdot \dots \cdot 1 = n!$ if $\#\Omega = n$.

Definition. A K-permutation of Ω is an ordered <u>subset</u> of K elements from Ω

no repeats.

Lemma. If $\#\Omega = n$, there are $(n)_K$ K-permutations

Factorials are hard to compute. Stirling's formula

$$n! \sim n^n e^{-n} \sqrt{2\pi n}$$

Proposition. $\lim_{n \to \infty} \frac{n^n e^{-n} \sqrt{2\pi n}}{n!} = 1.$

Mathematica demo.

Definition. A permutation ~~of~~ with $\sigma(\omega) = \omega$ has ω as a <u>fixed point</u>. A permutation with no fixed points is called a <u>derangement</u>.

Mathematica demo. $P(\text{derangement}) \to 1/e$.

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Length@PermutationSupport@RandomPermutation[n]==n.
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