Math 6500 Minihomework: Bernoulli numbers and polynomials

This minihomework covers our notes on the advanced error analysis of the trapezoid rule. We’ll make a lot of use of the two formulae:

\[
\frac{t}{e^t - 1} = \sum_{j=0}^{\infty} B_j \frac{t^j}{j!}
\]

and

\[
\frac{t(e^{xt} - 1)}{e^t - 1} = \sum_{j=0}^{\infty} B_j(x) \frac{t^j}{j!}.
\]

1. In the demonstration, we saw that the Bernoulli numbers and polynomials were related by the identity

\[B_i = -\int_0^1 B_i(x) \, dx\]

for \(i > 1\). We are now going to prove this in several steps.

a. Integrate both sides of (2) with respect to \(x\) from 0 to 1 to get a new series where the coefficients are integrals of \(B_j(x)\).

b. Prove using (1) that these integrals are actually the Bernoulli numbers \(B_j\), except at \(j = 0\).

2. In the notes, we gave as an exercise the task of proving that

\[\frac{d}{dx} B_j(x) = j B_{j-1}(x), \quad \text{for } j \geq 4, \text{ } j \text{ even}\]

and

\[\frac{d}{dx} B_j(x) = j (B_{j-1}(x) + B_{j-1}) \quad \text{for } j \geq 3, \text{ } j \text{ odd}.\]

Prove both of these formulae. Hint: Experiment with differentiating both sides of (2) with respect to \(x\).