Math/Csci 4690/6690 : Linear Algebra of Eigenvalues and Eigenvectors

In this minihomework, we recall some facts about eigenvectors and eigenvalues that you (hope-fully) covered in your linear algebra class.

1. (10 points) Suppose that $M$ is a matrix, and let $u$ and $v$ be vectors so that

$$Mu = \lambda u \quad \text{and} \quad Mv = \mu v.$$ 

Prove that if $M$ is symmetric and $\mu \neq \lambda$ then $\langle u, v \rangle = 0$.

Note: Make sure you use the fact that $M$ is symmetric— the statement is false otherwise!
2. (10 points)

**Definition.** An \( n \times n \) matrix \( Q \) is orthogonal if \( Q^T = Q^{-1} \).

We note that this means that \( QQ^T = Q^T Q = I_n \).

It is a fact that an orthogonal \( 2 \times 2 \) matrix is a rotation or reflection, and an orthogonal \( 3 \times 3 \) matrix is a rotation around some axis, possibly composed with a reflection in the plane normal to the axis of rotation.

For these reasons, we think of \( n \times n \) orthogonal matrices as generalized “rotations”, even though they may not have a single axis and angle.\(^1\)

(1) (10 points) Prove that if \( Q \) is orthogonal, then \( \langle u, v \rangle = \langle Qu, Qv \rangle \). In particular, we have \( \langle u, v \rangle = 0 \iff \langle Qu, Qv \rangle = 0 \). This is why we call these matrices “orthogonal”: they carry pairs of orthogonal vectors to pairs of orthogonal vectors.

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\(^1\)For example, an orthogonal matrix could act on \( \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 \) by rotating the two copies of \( \mathbb{R}^2 \) by two different angles.
(2) (10 points) Show that if $Mv = \lambda v$ and $Q$ is any orthogonal matrix then if $Mv = \lambda v$, we have $(QMQt)(Qv) = \lambda(Qv)$. That is, an orthogonal transformation of a matrix just rotates the eigenvectors; it doesn’t change the eigenvalues.
3. (15 points) A permutation of a vector \( v = (v_1, \ldots, v_n) \) is a rearrangement of its coordinates. For example \((v_2, v_1, v_3)\) is a permutation of \((v_1, v_2, v_3)\), as is \((v_3, v_2, v_1)\). We can represent a permutation by a bijective function

\[
\pi: \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}
\]

We can write the action of a permutation on a vector as a matrix:

\[
(\Pi)_{ij} = \begin{cases} 
1, & \text{if } \pi(i) = j, \\
0, & \text{otherwise}
\end{cases}
\]

For instance, the permutation \(\pi(1, 2, 3) = (2, 1, 3)\) is encoded by the matrix

\[
\Pi = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

as we can see by taking the product

\[
\Pi v = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3
\end{pmatrix}
= \begin{pmatrix}
v_2 \\
v_1 \\
v_3
\end{pmatrix}
\]

(1) (10 points) Prove that permutation matrices are orthogonal matrices.
(2) (5 points) Use the result above and the result of problem 1 to show that if $\Pi$ is a permutation matrix and $Mv = \lambda v$, we have

$$(\Pi M \Pi^T) (\Pi v) = \lambda (\Pi v)$$

That is, permuting the coordinates of a matrix just permutes the coordinates of the eigenvectors; it doesn’t change the eigenvalues.
4. (15 points)

**Definition.** Two $n \times n$ matrices $A$ and $B$ are similar if there is an invertible matrix $X$ so that $X^{-1}AX = B$.

Prove that if $A$ and $B$ are similar, then $A$ and $B$ have the same eigenvalues.
5. (15 points) (The Spectral Decomposition Theorem) Suppose that $M$ is an $n \times n$ symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and that $v_1, \ldots, v_n$ are a set of orthonormal column eigenvectors. Let $V$ be the (orthogonal) matrix whose $i$-th column is $v_i$. Prove

$$V^T M V = \Lambda,$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, the diagonal $n \times n$ matrix with $\lambda_1, \ldots, \lambda_n$ on the diagonal.

Note that this theorem implies that

$$M = V \Lambda V^T = \sum \lambda_i v_i v_i^T$$

where the $n \times n$ matrix $v_i v_i^T$ is the “outer product” of the column vector $v_i$ with itself. (The “inner product” or dot product is the $1 \times 1$ matrix $v_i^T v_i$.) This description of $M$ will often be helpful when $M$ is the diffusion operator or graph Laplacian!
6. (15 points)

**Definition.** The trace of an $n \times m$ matrix $M$ is given by the sum of diagonal entries:

$$
\text{tr } M := \sum M_{ii}
$$

It is a helpful fact that for any pair of matrices,

$$
\text{tr } AB = \text{tr } BA.
$$

Use this fact and the previous exercise to show that if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $M$,

$$
\text{tr } M = \lambda_1 + \cdots + \lambda_n.
$$