Math/Csci 4690/6690: Elementary properties of the graph Laplacian

In this minihomework, we will prove some of the basic properties of the graph Laplacian. First, recall from the notes that

**Definition.** The boundary map for a graph $G$ is the $v \times e$ matrix defined by

$$\partial(e_i) = \text{head}(e_i) - \text{tail}(e_i)$$

The graph Laplacian $L_G$ for a graph $G$ is the symmetric $v \times v$ matrix defined by

$$L_G := \partial \partial^T.$$ 

We proved in the notes that for any vector $v \in \mathbb{R}^v$, we have

$$Q_{L_G}(v) = \langle v, L_G v \rangle = \sum_{i=1}^{e} \left(v_{\text{head}(e_i)} - v_{\text{tail}(e_i)}\right)^2.$$ 

1. (10 points) Suppose that $G$ and $G'$ have the same vertices and edges $v_1, \ldots, v_v = v'_1, \ldots, v'_v$ and $e_1, \ldots, e_e = e'_1, \ldots, e'_e$, but the orientations of the edges may not agree.
   Prove that $L_G = L_{G'}$. 


2. (10 points) Recall that

**Definition.** The degree matrix $D_G$ is the diagonal matrix whose entries are the degrees of the vertices $v_1, \ldots, v_n$, and the adjacency matrix $M_G$ of $G$ is the symmetric matrix defined by

$$(M_G)_{ij} = \begin{cases} 
1, & \text{if } v_i \text{ and } v_j \text{ are joined by an edge} \\
0, & \text{if not.} 
\end{cases}$$

3. (10 points) Let $\lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of the symmetric matrix $L_G$.

(1) (5 points) Prove that $\lambda_1 = 0$ by finding a vector $v \in \mathbb{R}^n$ so that $L_G v = 0$. 

(2) (5 points)

**Definition.** A set $S$ of vertices of $G$ is called a component of $G$ if there are no edges joining vertices in $S$ to vertices outside $S$. The largest set of disjoint components $S_1, \ldots, S_d$ of $G$ are called the connected components of $G$.

Prove that if $G$ has $d$ connected components, then $\lambda_1 = \lambda_2 = \ldots = \lambda_d = 0$.

Hint: The $S_i$ really partition $G$ into $d$ separate graphs $G_1, \ldots, G_d$. 