Math 4500/6500 Minihomework: Richardson Extrapolation

This homework assignment covers our notes on Richardson Extrapolation. If you find the algebra lengthy or irritating (which is pretty likely), you are encouraged to use *Mathematica* to do it.

1. Use the Taylor expansion of $f(x)$ to find an expression of the form

$$f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] + E_1$$

for $f'(x)$ where E_1 is some error term. This approximation for $f'(x)$ uses two points spaced at distance $2h$, just like the central approximation to the derivative

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + E_2.$$

which we learned in class. Compare the error terms E_1 and E_2. Which one is better?

2. Use Taylor's Theorem to prove that for any $h > 0$, if f is smooth (has continuous derivatives of every order) on $[x, x+2h]$ then

$$f'(x) - \frac{1}{2h}[4f(x+h) - 3f(x) - f(x+2h)] = \frac{1}{3}h^2 f'''(x) + \text{higher order terms in } h$$

3. Following the model in the notebook `http://www.jasoncantarella.com/downloads/symbolic_richardson_extrapolation.nb`, use *Mathematica* to find the Richardson extrapolation formulae for the third derivative.

   a. Test these formulae for the third derivative of cosine at $x = 2.0$ by choosing some point (or points) to evaluate your formulae (with various values of h) and comparing your results to the true value of the third derivative of cosine.

   b. What is the largest number of correct digits you can get by choosing different values of h?

   c. How does this compare to the largest number of correct digits you can get for the Richardson extrapolation for the *first* derivative of cosine by choosing h carefully?