Definition. The adjacency matrix of a graph $G$, denoted $M_G$, is the symmetric matrix where

$$(M_G)_{ij} = \begin{cases} 
1, & \text{if } v_i \leadsto v_j \text{ is an edge of } G \\
0, & \text{if not.}
\end{cases}$$

We denote the eigenvalues of $M_G$ by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_\nu$.

Theorem. If $S$ is any subgraph of $G$, $d_{\text{ave}}(S)$ is the average degree of vertices in the subgraph, and $d_{\text{max}}$ is the maximum degree of any vertex in $G$, then

$$d_{\text{ave}}(S) \leq \mu_1 \leq d_{\text{max}}$$

Theorem. If $G$ is connected and $\mu_1 = d_{\text{max}}$, then $G$ is $d_{\text{max}}$-regular. If $G$ is $d$-regular, then $\mu_1 = d$.

Definition. A coloring of a graph is an assignment of colors to vertices so that every edge joins vertices of different colors. A graph is $k$-colorable if a coloring exists with $k$ colors. The chromatic number $\chi(G)$ of a graph is the smallest $k$ for which $G$ is $k$-colorable.

Theorem. For any graph $G$, $\chi(G) \leq \lceil \mu_1 \rceil + 1$.

Theorem. If $\mu_1 = -\mu_\nu$ then $G$ is 2-colorable. If $G$ is 2-colorable, then the eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_\nu$ are symmetric around 0.

1. (20 points) Consider the graph

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1. Every vertex has degree $d$.
2. A graph which is 2-colorable is said to be bipartite.
(1) (10 points) Establish an upper bound on $\chi(G)$ by finding a coloring of $G$ with as few colors as possible.
(2) (10 points) Prove that your upper bound is actually equal to $\chi(G)$ by using our theorems to show that no coloring with fewer colors exists. (If you need eigenvalues of $M_G$, it’s expected that you’ll use a computer to find them. This is an acceptable proof technique as long as you include screenshots.)
2. (10 points) Consider the graph

Find the best bounds on $\chi(G)$ that you can by explicitly finding colorings and using our theorems above. Can you compute $\chi(G)$ exactly?
3. (10 points) Consider the graph

Find as many eigenvalues of $M_G$ as you can \textit{without} using a computer; prove that each number you give is actually an eigenvalue of $G$. 