Math 4250/6250 Mini-Homework

This minihomework accompanies the lecture notes on “The Second Fundamental Form (Part II)”. In this minihomework, we’ll go through a series of exercises on the relationship between linear maps, quadratic forms, matrices, and symmetry. We’ll need some definitions:

**Definition 1.** Suppose that $M : V \rightarrow V$. We say that $M$ is a linear map if for any $\vec{x}, \vec{y} \in V$ and $a, b \in \mathbb{R}$ we have

$$M(a\vec{x} + b\vec{y}) = aM(\vec{x}) + bM(\vec{y}).$$

**Definition 2.** Suppose that $V$ is a (finite-dimensional) real vector space. We say that $Q : V \rightarrow \mathbb{R}$ is a quadratic form if

$$Q(a\vec{v}) = a^2Q(\vec{v}) \quad \text{and} \quad Q(\vec{u}, \vec{v}) = \frac{1}{2}(Q(\vec{u} + \vec{v}) - Q(\vec{u}) - Q(\vec{v}))$$

is bilinear.

**Definition 3.** Suppose that $V$ is a (finite-dimensional) real vector space, $M : V \rightarrow V$ is a linear map, and $Q : V \rightarrow \mathbb{R}$ is a quadratic form. We say that $M$ is symmetric with respect to $Q$ if for any $\vec{x}, \vec{y} \in V$ we have

$$Q(\vec{x}, M(\vec{y})) = Q(M(\vec{x}), \vec{y}).$$

It is common to blur the distinction between linear maps and matrices and also between quadratic forms and the symmetric matrices which represent them. In this minihomework, you’ll go back to basics in order to understand the difference between a linear map which is represented by a symmetric matrix (in a given basis) and a linear map which is symmetric with respect to a particular quadratic form.

1. Using Definition 2,

   a. Prove that $Q(\vec{u}, \vec{v}) = Q(\vec{v}, \vec{u})$.

   b. Suppose $\vec{v}_1, \ldots, \vec{v}_n$ is any basis for $V$, $Q(\vec{v}_i, \vec{v}_j) = q_{ij}$, and $\vec{x} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$. Prove

   $$Q(\vec{x}) = Q(\vec{x}, \vec{x}) = \sum_{i,j} q_{ij}x_ix_j$$

   and that $q_{ij} = q_{ji}$.

   c. Suppose $\langle \cdot, \cdot \rangle$ is the ordinary dot product on $\mathbb{R}^n$, and we have vectors $\vec{x} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$ and $\vec{y} = y_1\vec{v}_1 + \cdots + y_n\vec{v}_n$ in $V$. Show that for the $q_{ij}$ from part b, we have

   $Q(\vec{x}, \vec{y}) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$. \hfill (1)

   1Because of this relationship, we often refer interchangeably to the quadratic form $Q(\vec{u}, \vec{v})$ and the associated symmetric $n \times n$ matrix $(q_{ij})$. However, we should keep in mind that the matrix $(q_{ij})$ depends on the basis $\vec{v}_1, \ldots, \vec{v}_n$ we choose for $V$, but the quadratic form $Q$ does not.
2. Suppose that $\vec{v}_1, \ldots, \vec{v}_n$ is any basis for $V$, and $M: V \rightarrow V$ is a linear map.

a. Show from Definition [1] that there are some $m_{ij} \in \mathbb{R}$ so that
\[
M(\vec{v}_j) = \sum_{i=1}^{n} m_{ij} \vec{v}_i
\]  
(2)

b. Show from Definition [1] and (2) that if $\vec{x} = x_1 \vec{v}_1 + \cdots + x_n \vec{v}_n$ then
\[
M(\vec{x}) = \sum_{j,i=1}^{n} m_{ij} x_j \vec{v}_i
\]  
(3)

c. Suppose that $\vec{x} = x_1 \vec{v}_1 + \cdots + x_n \vec{v}_n$, the $m_{ij}$ are as defined in part b, and $M(\vec{x}) = \vec{y} = y_1 \vec{v}_1 + \cdots + y_n \vec{v}_n$. Show that
\[
\begin{pmatrix}
   y_1 \\
   \vdots \\
   y_n
\end{pmatrix} =
\begin{pmatrix}
   m_{11} & \cdots & m_{1n} \\
   \vdots & \ddots & \vdots \\
   m_{n1} & \cdots & m_{nn}
\end{pmatrix}
\begin{pmatrix}
   x_1 \\
   \vdots \\
   x_n
\end{pmatrix}.
\]  
(4)

3. Now we’re going to connect quadratic forms, linear maps, matrices and symmetry.

a. Prove the following statement:

Suppose $M$ is a linear map $V \rightarrow V$ and $Q$ is a quadratic form on $V$. Further, suppose that $\vec{v}_1, \ldots, \vec{v}_n$ is a basis for $V$ and with respect to this basis, $Q$ has matrix $(q_{ij})$, and $M$ has matrix $(m_{ij})$. If $(q_{ij})$ is a diagonal matrix and $M$ is symmetric with respect to $Q$, then $(m_{ij})$ is a symmetric matrix.

b. Show that we can’t remove the hypothesis “$(q_{ij})$ is a diagonal matrix” by considering
\[
(q_{ij}) = \begin{pmatrix}
   1 & 2 \\
   2 & 3
\end{pmatrix} \quad (m_{ij}) = \begin{pmatrix}
   1 & 3 \\
   1 & 1
\end{pmatrix}.
\]

and proving that the linear map $M$ given by $(m_{ij})$ is symmetric with respect to the quadratic form $Q$ given by $(q_{ij})$.

---

2 We often use $M$ to denote both the linear map and the $n \times n$ matrix $(m_{ij})$ because of the close relationship between the two objects given by (4). However, as with $Q$ and $(q_{ij})$, we should remember that the matrix $(m_{ij})$ depends on the basis $\vec{v}_1, \ldots, \vec{v}_n$ while the linear map $M$ does not. Further, notice that unlike the matrix $(q_{ij})$, which is always symmetric, the matrix $(m_{ij})$ may or may not be symmetric.

3 This means that $Q(\vec{v}_i, \vec{v}_j) = 0$ when $i \neq j$. In this case we say that $\vec{v}_1, \ldots, \vec{v}_n$ is a $Q$-orthogonal basis for $V$. 
