The Biot–Savart operator for application to knot theory, fluid dynamics, and plasma physics

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The *writhing number* of a curve in 3-space is the standard measure of the extent to which the curve wraps and coils around itself; it has proved its importance for molecular biologists in the study of knotted DNA and of the enzymes which affect it. The *helicity* of a vector field defined on a domain in 3-space is the standard measure of the extent to which the field lines wrap and coil around one another; it plays important roles in fluid dynamics and plasma physics. The *Biot–Savart operator* associates with each current distribution on a given domain the restriction of its magnetic field to that domain. When the domain is simply connected, the divergence-free fields which are tangent to the boundary and which minimize energy for given helicity provide models for stable force-free magnetic fields in space and laboratory plasmas; these fields appear mathematically as the extreme eigenfields for an appropriate modification of the Biot–Savart operator. Information about these fields can be converted into bounds on the writhing number of a given piece of DNA. The purpose of this paper is to reveal new properties of the Biot–Savart operator which are useful in these applications. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1329659]

I. INTRODUCTION

Let Ω be a compact domain with smooth boundary in 3-space, and let $VF(\Omega)$ be the space of smooth vector fields on Ω with the L^2 inner product $\langle V,W \rangle = \int_\Omega V \cdot W \, d(\text{vol})$. By ''smooth,'' equivalently C^∞, we mean that derivatives of all orders exist and are continuous.

If we think of the smooth vector field V on Ω as a distribution of electric current, then the Biot–Savart formula

$$\text{BS}(V)(y) = (1/4\pi) \int_\Omega V(x) \times (y-x)/|y-x|^3 \ d(\text{vol}_x)$$

gives the resulting magnetic field $\text{BS}(V)$ throughout 3-space. If we restrict this magnetic field to the domain Ω, then we get the *Biot–Savart operator*,

$$\text{BS}:VF(\Omega) \rightarrow VF(\Omega).$$

Theorem A: *The equation* $\nabla \times \text{BS}(V) = V$ *holds in* Ω *if and only if* V *is divergence-free and tangent to the boundary of* Ω.

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It is well known that curl is a left inverse to the Biot–Savart operator when the input field V is divergence-free and tangent to the boundary. The new information is that this result holds in no other cases. The impact of this is that eigenvalue problems for the Biot–Savart operator, which are central to the study of helicity, cannot in general be converted to eigenvalue problems for curl (that is, to a system of partial differential equations).

Theorem B: *The kernel of the Biot–Savart operator is precisely the space of gradient vector fields which are orthogonal to the boundary of* Ω.

Actually, somewhat more is true. If V is a smooth gradient vector field defined on Ω and orthogonal to its boundary, then its magnetic field $BS(V)=0$ throughout 3-space. Conversely, if V is a smooth vector field defined on Ω whose magnetic field $BS(V)=0$ in Ω, then V is a gradient field orthogonal to the boundary of Ω, and hence $BS(V)=0$ throughout 3-space.

Theorem C: *The image of the Biot–Savart operator is a proper subspace of the image of curl, whose orthogonal projection into the subspace of ''fluxless knots'' is one-to-one.*

Vector fields on the domain Ω which are divergence-free and tangent to its boundary are called *fluid knots*; we explain this terminology in Sec. IV. *Fluxless knots* are fluid knots with zero flux through every cross-sectional surface $(\Sigma,\partial\Sigma)\subset(\Omega,\partial\Omega)$. The above theorems lead to several interesting examples of ''impossible'' magnetic fields. Nevertheless, Theorem C falls short of giving a precise characterization of the image of the Biot–Savart operator, and hence of those fields in a domain Ω which are magnetic fields of current distributions within Ω.

Theorem D: *The Biot–Savart operator is a bounded operator, and hence extends to a bounded operator on the L^2 completion of its domain, where it is both compact and self-adjoint.*

The eigenfields of this operator which correspond to its extreme eigenvalues turn out to be the vector fields in Ω with minimum energy for given helicity. If we start with a vector field V which is divergence-free and tangent to the boundary of its domain Ω, that is, a fluid knot, then its magnetic field $BS(V)$, though divergence-free, will in general not be tangent to the boundary of Ω. In such a case, we simply modify the Biot–Savart operator BS by following it by orthogonal projection back to the subspace of fluid knots. The eigenfields of this modified Biot–Savart operator which correspond to its extreme eigenvalues are then the fluid knots in Ω with minimum energy for given helicity. When the domain Ω is simply connected, these energy-minimizers model the stable plasma fields in Ω.

II. PRELIMINARIES

A. Writhing, helicity, and the Biot–Savart operator

The *writhing number* $\mathrm{Wr}(K)$ of a smooth curve K in 3-space, defined by the formula

$$\mathrm{Wr}(K)=(1/4\pi)\int_{K\times K}(dx/ds\times dy/dt)\cdot(x-y)/|x-y|^3 \ ds \ dt, \qquad (2.1)$$

was introduced by Călugăreanu[1–3] in 1959–1961 and named by Fuller[4] in 1971, and is the standard measure of the extent to which the curve wraps and coils around itself.

The *helicity* $H(V)$ of a smooth vector field V on the domain Ω in 3-space, defined by the formula

$$H(V)=(1/4\pi)\int_{\Omega\times\Omega}V(x)\times V(y)\cdot(x-y)/|x-y|^3 \ d(\mathrm{vol}_x) \ d(\mathrm{vol}_y), \qquad (2.2)$$

was introduced by Woltjer[5] in 1958 and named by Moffatt[6] in 1969, and is the standard measure of the extent to which the field lines wrap and coil around one another.

Clearly, helicity for vector fields is the analogue of writhing number for knots.

The helicity of V is closely related to its image under the Biot–Savart operator,

$$H(V) = (1/4\pi) \int_{\Omega \times \Omega} V(x) \times V(y) \cdot (x-y)/|x-y|^3 \ d(\text{vol}_x) \ d(\text{vol}_y)$$

$$= \int_{\Omega} V(y) \cdot \left[(1/4\pi) \int_{\Omega} V(x) \times (y-x)/|y-x|^3 \ d(\text{vol}_x) \right] d(\text{vol}_y)$$

$$= \int_{\Omega} V(y) \cdot \text{BS}(V)(y) \ d(\text{vol}_y)$$

$$= \int_{\Omega} V \cdot \text{BS}(V) \ d(\text{vol}),$$

so the helicity of V is just the L^2 inner product of V and $\text{BS}(V)$,

$$H(V) = \langle V, \text{BS}(V) \rangle. \tag{2.3}$$

It is because of this formula that the Biot–Savart operator,

$$\text{BS}:\text{VF}(\Omega) \rightarrow \text{VF}(\Omega), \tag{2.4}$$

plays such a prominent role in the study of writhing of knots and helicity of vector fields.

B. Applications: A quick guide to the literature

For a glance at the prehistory of the writhing number, see Gauss's half-page note[7] (1833) on an integral formula for the linking number of two disjoint closed curves in 3-space. Rewrite his expression in modern notation and let the two curves coincide and you will have the formula for the writhing number.

The writhing number has proved its importance for molecular biologists in the study of knotted duplex DNA and of the enzymes which affect it; see White,[8] Fuller,[9] Bauer, Crick, and White,[10] Wang,[11] Sumners,[12–14] and Cantarella, Kusner, and Sullivan.[15]

For an overview of the connection between knot theory and electrodynamics, see Lomonaco.[16]

Woltjer's formula for the helicity of a vector field arose from his interest in force-free magnetic fields. These are magnetic fields which are everywhere parallel to the current flows which give rise to them, so that the Lorentz force on the flowing charged particles is zero. Because the gross magnetic field in the Crab Nebula appeared to be steady over a number of years, Woltjer believed it to be force-free, and studied[17] it in great detail. Two early papers on force-free magnetic fields are Lundquist[18] and Chandrasekhar–Kendall.[19] Two more recent papers are Laurence and Avellaneda[20] and Tsuji.[21] Marsh's book[22] has an extensive and up-to-date bibliography on this subject.

For a study of the connection between writhing and helicity, see Berger and Field[23] and Moffatt and Ricca.[24,25]

For the connection between helicity and the ordinary and asymptotic Hopf invariants, see Whitehead[26] and Arnold.[27]

For an introduction to the spectral theory of the Biot–Savart operator and its use in determining upper bounds for writhing and helicity, see Ref. 28. For explicit computation of extreme eigenfields, see Refs. 29 and 30. For an analysis of isoperimetric problems connected with the Biot–Savart operator, see Ref. 31. For application to the qualitative study of stable plasma flows, see Cantarella.[32] For an overview of our work, see our survey paper.[33]

For further information on related spectral problems for the curl operator, see Yoshida and Giga.[34]

For the connection between this spectral theory and plasma physics, see Yoshida.[35]

For a study of magnetic field generation in electrically conducting fluids, see the book by Moffatt.[36]

For connections with dynamo theory, see the survey article by Childress.[37]

For many papers on the connections with the dynamics of fluids and plasmas, see the books by Moffatt and Tsinober[38] and by Moffatt, Zaslavsky, Comte, and Tabor.[39]

For the connections between force-free fields, contact topology and fluid dynamics, see Etnyre and Ghrist.[40]

C. The Hodge decomposition theorem

In this section we present the Hodge Decomposition Theorem for vector fields on bounded domains in R^3, which we will use throughout the paper. Although we state it below for the space $VF(\Omega)$ of smooth vector fields on Ω with the usual L^2 inner product, it holds just as well for the L^2 completions of $VF(\Omega)$ and of the various subspaces described below.

The papers of Weyl[41] and Friedrichs,[42] the notes of Blank, Friedrichs, and Grad,[43] and the book of Schwarz[44] are all good references; an exposition of this theorem in the form given below appears in Ref. 45.

Hodge Decomposition Theorem: *We have a direct sum decomposition of* $VF(\Omega)$ *into five mutually orthogonal subspaces,*

$$VF(\Omega) = FK \oplus HK \oplus CG \oplus HG \oplus GG,$$

with

$$\ker\ \mathrm{curl} = HK \oplus CG \oplus HG \oplus GG,$$

$$\mathrm{image}\ \mathrm{grad} = CG \oplus HG \oplus GG,$$

$$\mathrm{image}\ \mathrm{curl} = FK \oplus HK \oplus CG,$$

$$\ker\ \mathrm{div} = FK \oplus HK \oplus CG \oplus HG,$$

where

$FK = $ Fluxless Knots $= \{\nabla \cdot V = 0,\ V \cdot n = 0,\ \text{all interior fluxes} = 0\},$

$HK = $ Harmonic Knots $= \{\nabla \cdot V = 0,\ \nabla \times V = 0,\ V \cdot n = 0\},$

$CG = $ Curly Gradients $= \{V = \nabla \varphi,\ \nabla \cdot V = 0,\ \text{all boundary fluxes} = 0\},$

$HG = $ Harmonic Gradients $= \{V = \nabla \varphi,\ \nabla \cdot V = 0,\ \varphi \ \text{locally constant on}\ \partial\Omega\},$

$GG = $ Grounded Gradients $= \{V = \nabla \varphi,\ \varphi|_{\partial\Omega} = 0\},$

and furthermore,

$$HK \cong H_1(\Omega;R) \cong H_2(\Omega, \partial\Omega;R) \cong R^{\text{genus of } \partial\Omega},$$

$$HG \cong H_2(\Omega;R) \cong H_1(\Omega, \partial\Omega;R) \cong R^{(\#\ \text{components of}\ \partial\Omega) - (\#\ \text{components of}\ \Omega)}.$$

We need to explain the meanings of the conditions which appear in the statement of this theorem.

The outward pointing unit vector field orthogonal to $\partial\Omega$ is denoted by n, so the condition $V \cdot n = 0$ indicates that the vector field V is tangent to the boundary of Ω.

Let Σ stand generically for any smooth surface in Ω with $\partial\Sigma\subset\partial\Omega$. Earlier, when commenting on the statement of Theorem C, we indicated this by writing $(\Sigma,\partial\Sigma)\subset(\Omega,\partial\Omega)$. Now, orient Σ by picking one of its two unit normal vector fields n. Then, for any vector field V on Ω, we can define the *flux* of V through Σ to be the value of the integral $\Phi=\int_\Sigma V\cdot n\ \mathrm{d(area)}$.

Assume that V is divergence-free and tangent to $\partial\Omega$. Then the value of this flux depends only on the homology class of Σ in the relative homology group $H_2(\Omega,\partial\Omega;Z)$. For example, if Ω is an n-holed solid torus, then there are disjoint oriented cross-sectional disks $\Sigma_1,\ldots,\Sigma_n$, positioned so that cutting Ω along these disks will produce a simply-connected region. The fluxes $\Phi_1,\ldots,\Phi_n$ of V through these disks determine the flux of V through any other cross-sectional surface.

If the flux of V through every smooth surface Σ in Ω with $\partial\Sigma\subset\partial\Omega$ vanishes, we say "*all interior fluxes*=0." Then,

$$\mathrm{FK}=\{V\epsilon\ \mathrm{VF}(\Omega):\nabla\cdot V=0,\ V\cdot n=0,\ \text{all interior fluxes}=0\} \tag{2.5}$$

will be the subspace of *fluxless knots*, already mentioned when explaining the statement of Theorem C.

The subspace,

$$\mathrm{HK}=\{V\epsilon\ \mathrm{VF}(\Omega):\nabla\cdot V=0,\ \nabla\times V=0,\ V\cdot n=0\} \tag{2.6}$$

of *harmonic knots* is isomorphic to the absolute homology group $H_1(\Omega;R)$ and also, via Poincaré duality, to the relative homology group $H_2(\Omega,\partial\Omega;R)$, and is thus a finite-dimensional vector space, with dimension equal to the genus of $\partial\Omega$.

The orthogonal direct sum of these two subspaces,

$$\mathrm{K}(\Omega)=\mathrm{FK}\oplus\mathrm{HK} \tag{2.7}$$

is the subspace of $\mathrm{VF}(\Omega)$ consisting of all divergence-free vector fields defined on Ω and tangent to its boundary. These are the vector fields that represent current flows in the standard versions of the laws of Magnetostatics.

We called these vector fields *fluid knots* in the Introduction, and pause to explain this terminology. Given a knot in 3-space, we can choose a thin tubular neighborhood of the knot to be our domain Ω, and then choose a divergence-free vector field V in Ω, for example orthogonal to the cross-sectional disks and hence tangent to the boundary. In this way, questions about the geometry of the knot can sometimes profitably be reformulated as questions about the vector field V, our "fluid knot." We did exactly this in our paper[28] when deriving an upper bound for the writhing number of a knot of given length and thickness.

If V is a vector field defined on Ω, we will say that *all boundary fluxes of V are zero* if the flux of V through each component of $\partial\Omega$ is zero. Then,

$$\mathrm{CG}=\{V\epsilon\ \mathrm{VF}(\Omega):V=\nabla\varphi,\ \nabla\cdot V=0,\ \text{all boundary fluxes}=0\} \tag{2.8}$$

will be called the subspace of *curly gradients* because these are the only gradients which lie in the image of curl.

Next we define the subspace of *harmonic gradients*,

$$\mathrm{HG}=\{V\epsilon\ \mathrm{VF}(\Omega):V=\nabla\varphi,\ \nabla\cdot V=0,\ \varphi\ \text{locally constant on}\ \partial\Omega\}, \tag{2.9}$$

meaning that φ is constant on each component of $\partial\Omega$. This subspace is isomorphic to the absolute homology group $H_2(\Omega;R)$ and also, via Poincaré duality, to the relative homology group $H_1(\Omega,\partial\Omega;R)$, and is hence a finite-dimensional vector space, with dimension equal to the number of components of $\partial\Omega$ minus the number of components of Ω.

The definition of the subspace of *grounded gradients*,

$$\mathrm{GG}=\{V\epsilon\ \mathrm{VF}(\Omega):V=\nabla\varphi,\ \varphi|_{\partial\Omega}=0\}, \tag{2.10}$$

is self-explanatory.

A vector field V belongs to the subspace $HG \oplus GG$ of $VF(\Omega)$ if and only if it is the gradient of a smooth function φ on Ω which is constant on each component of $\partial\Omega$, or equivalently, is a gradient vector field which is orthogonal to $\partial\Omega$. Theorem B asserts that these vector fields form the kernel of the Biot–Savart operator.

The five orthogonal direct summands of $VF(\Omega)$ can be characterized as follows:

$$FK = (\ker \text{ curl})^\perp,$$

$$HK = (\ker \text{ curl}) \cap (\text{image grad})^\perp,$$

$$CG = (\text{image grad}) \cap (\text{image curl}),$$

$$HG = (\ker \text{ div}) \cap (\text{image curl})^\perp,$$

$$GG = (\ker \text{ div})^\perp.$$

These characterizations bear witness to the geometric and analytic significance of the summands.

We end this section with examples of vector fields from each of the five summands.

1. FK=fluxless knots

Let Ω be a round ball of radius 1, centered at the origin in 3-space. Consider the vector field

$$V = -y\hat{\imath} + x\hat{\jmath}.$$

This is the velocity field for rotation of 3-space about the z-axis at constant angular speed. It is divergence-free and tangent to the boundary of the ball Ω, and hence belongs to the subspace FK of fluxless knots, because there are no harmonic knots on a ball.

2. HK=harmonic knots

Let Ω be a solid torus of revolution about the z-axis. Using cylindrical coordinates (r, φ, z), consider the vector field

$$V = (1/r)\hat{\varphi},$$

which is the magnetic field due to a steady current running up the z-axis. It is divergence-free and curl-free and tangent to the boundary of the solid torus Ω, and hence belongs to the subspace HK of harmonic knots.

3. CG=curly gradients

Let Ω be a round ball of radius 1, centered at the origin. Consider the harmonic function z, and the gradient field

$$V = \nabla z = \hat{k}.$$

This vector field is divergence-free and has zero flux through the one and only component of $\partial\Omega$, hence it belongs to the subspace CG of curly gradients.

4. HG=harmonic gradients

Let Ω be the region between two concentric round spheres, say of radius 1 and 2, centered at the origin. Using spherical coordinates (r, θ, φ), consider the harmonic function $1/r$, and its gradient vector field

$$V = \nabla(1/r) = (-1/r^2)\hat{r},$$

just the inverse square central field. Since the harmonic function $1/r$ is constant on each component of $\partial\Omega$, the vector field V belongs to the subspace HG of harmonic gradients. We may think of V as the electric field between two concentric spheres held at different potentials.

5. GG=grounded gradients

Let Ω be a round ball of radius 1, centered at the origin. Consider the function given by $r^2-1=x^2+y^2+z^2-1$, and the vector field

$$V=\nabla(r^2-1)=2x\hat{\imath}+2y\hat{\jmath}+2z\hat{k}.$$

Since the function r^2-1 has constant value zero on the boundary of Ω, the vector field V belongs to the subspace GG of grounded gradients. We may view V as an electric field with interior charges inside a conducting boundary.

III. STANDARD INFORMATION ABOUT THE BIOT–SAVART OPERATOR

A. The basic facts

Given a smooth vector field V on Ω, the *vector potential* A(V) for BS(V) is defined by the formula,

$$A(V)(y)=(1/4\pi)\int_\Omega V(x)/|y-x| \ \ d(\text{vol}_x). \tag{3.1}$$

Here is the classically known information about the Biot–Savart operator and its vector potential. Note that some of the assertions below hold for any vector field $V\epsilon\text{VF}(\Omega)$, while others need the more restrictive assumption that V is divergence-free and tangent to the boundary of Ω, in other words, that V lies in the subspace K(Ω) of fluid knots.

Standard Information: *Let Ω be a compact domain in 3-space with smooth boundary $\partial\Omega$. Let V be a smooth vector field defined on Ω. Then*

(1) BS(V) *and* A(V) *are well-defined on all of 3-space, that is, the improper integrals defining them converge everywhere;*
(2) BS(V) *and* A(V) *are of class* C^∞ *on* Ω*, and on the closure* Ω' *of* $R^3-\Omega$*.* BS(V) *is continuous on* R^3*, but its derivatives typically suffer jump discontinuities as one crosses* $\partial\Omega$*.* A(V) *is of class* C^1 *on* R^3*, but its second derivatives typically suffer jump discontinuities as one crosses* $\partial\Omega$*;*
(3) $\Delta A(V)=-V$ *in* Ω *and* $\Delta A(V)=0$ *in* Ω'*, where* Δ *is the vector Laplacian;*
(4) $\nabla\times A(V)=BS(V)$ *on* R^3*;*
(5) If $V\epsilon K(\Omega)$*, then* A(V) *is divergence-free on* R^3*;*
(6) $\nabla\cdot BS(V)=0$ *in* Ω *and in* Ω'*;*
(7) If $V\epsilon K(\Omega)$*, then* $\nabla\times BS(V)=V$ *in* Ω *and* $\nabla\times BS(V)=0$ *in* Ω'*;*
(8) If $V\epsilon K(\Omega)$*, then* $\int_C BS(V)\cdot ds=0$ *for all closed curves* C *on* $\partial\Omega$ *which bound in* $R^3-\Omega$*;*
(9) In general, A(V) *decays at* ∞ *like* $1/r$ *and* BS(V) *decays at* ∞ *like* $1/r^2$*; however, if* $V\epsilon K(\Omega)$*, then* A(V) *decays at* ∞ *like* $1/r^2$ *and* BS(V) *decays at* ∞ *like* $1/r^3$*.*

Proofs of most of these basic facts can be found throughout the physics literature (see, for example, Griffiths[46]), with the exception of item (9), which we prove in the Appendix. Item (7) contains the first half of Theorem A; we will prove that immediately, since it affects the rest of the paper.

B. Proof of (7)

The argument to follow begins as in Griffiths,[46] pp. 215–217, but is then modified to suit our purpose.

To prove (7), we assume that V is a fluid knot, and must show that

$$\nabla_y \times \text{BS}(V)(y) = V(y), \quad \text{when } y \epsilon \Omega,$$

$$= 0, \quad \text{when } y \epsilon \Omega'.$$

From now on, we will use the shorthand notation $\{V(y) \text{ in } \Omega / 0 \text{ in } \Omega'\}$, or simply $\{V(y)/0\}$, to express these two outcomes.

The above assertion will follow immediately from the next proposition, which will then serve as a springboard to the rest of the paper.

Proposition 1:

$$\nabla_y \times \text{BS}(V)(y) = \{V(y) \text{ in } \Omega / 0 \text{ in } \Omega'\} + (1/4\pi)\nabla_y \int_\Omega (\nabla_x \cdot V(x))/|y-x| \ d(\text{vol}_x)$$

$$- (1/4\pi)\nabla_y \int_{\partial\Omega} V(x) \cdot n/|y-x| \ d(\text{area}_x).$$

If V is divergence-free, then the second term on the right-hand side vanishes; if V is tangent to the boundary of Ω, then the third term on the right-hand side vanishes. If both hold, that is, if V is a fluid knot, then we get item (7).

We can view the statement of Proposition 1 as Maxwell's equation,

$$\nabla \times B = J + \partial E/\partial t, \tag{3.2}$$

as follows.

Let V represent a current distribution throughout the domain Ω. At time $t=0$, let the volume charge density ρ throughout Ω and the surface charge density σ along $\partial\Omega$ both be zero. Then set

$$\rho = -(\nabla \cdot V)t \quad \text{throughout } \Omega, \tag{3.3}$$

and

$$\sigma = (V \cdot n)t \quad \text{along } \partial\Omega. \tag{3.4}$$

Equation (3.3) for the volume charge density ρ is forced on us by the continuity equation,

$$\nabla \cdot V = -\partial\rho/\partial t. \tag{3.5}$$

Likewise, Eq. (3.4) for the surface charge density σ is forced on us by a version of the continuity equation appropriate to the boundary of our domain. The current V is simply carrying charge from locations within Ω and on its boundary to other such locations. Thus the surface charge density given by (3.4) has a time rate of change equal to the flux density of the current V through the boundary $\partial\Omega$.

Now the volume charge throughout Ω gives rise to a time varying electric field

$$E_\rho(y,t) = \left[(1/4\pi)\nabla_y \int_\Omega (\nabla_x \cdot V(x))/|y-x| \ d(\text{vol}_x) \right] t, \tag{3.6}$$

and the surface charge along $\partial\Omega$ gives rise to a time varying electric field,

$$E_\sigma(y,t) = \left[-(1/4\pi)\nabla_y \int_{\partial\Omega} V(x) \cdot n(x)/|y-x| \ d(\text{area}_x) \right] t, \tag{3.7}$$

both fields extending throughout 3-space.

The total electric field

$$E(y,t) = E_\rho(y,t) + E_\sigma(y,t) \tag{3.8}$$

has a time rate of change

$$\partial E/\partial t = \partial E_\rho/\partial t + \partial E_\sigma/\partial t = E'_\rho + E'_\sigma. \tag{3.9}$$

With this notation, the equation of Proposition 1 condenses to

$$\nabla \times \mathrm{BS}(V) = \{V/0\} + E'_\rho + E'_\sigma, \tag{3.10}$$

which is just Maxwell's Eq. (3.2). Proving Proposition 1 confirms these interpretations.

Proof of Proposition 1: We must evaluate

$$\nabla_y \times \mathrm{BS}(V)(y) = \nabla_y \times (1/4\pi) \int_\Omega V(x) \times (y-x)/|y-x|^3 \ d(\mathrm{vol}_x)$$

$$= (1/4\pi) \int_\Omega \nabla_y \times \{V(x) \times (y-x)/|(y-x)|^3\} \ d(\mathrm{vol}_x). \tag{3.11}$$

We will need the following formula from vector calculus:

$$\nabla \times (A \times B) = (B \cdot \nabla)A - (A \cdot \nabla)B + A(\nabla \cdot B) - B(\nabla \cdot A). \tag{3.12}$$

Applying this formula to the integrand, we get

$$\nabla_y \times \{V(x) \times (y-x)/|y-x|^3\}$$
$$= ((y-x)/|y-x|^3 \cdot \nabla_y)V(x) - (V(x) \cdot \nabla_y)((y-x)/|y-x|^3)$$
$$+ V(x)\nabla_y \cdot ((y-x)/|y-x|^3) - ((y-x)/|y-x|^3)(\nabla_y \cdot V(x)). \tag{3.13}$$

The first and last terms on the right-hand side are zero, because they involve differentiation with respect to y of $V(x)$, which depends only on x. Thus,

$$\nabla_y \times \{V(x) \times (y-x)/|y-x|^3\} = V(x)\nabla_y \cdot ((y-x)/|y-x|^3) - (V(x) \cdot \nabla_y)((y-x)/|y-x|^3). \tag{3.14}$$

In the first term on the right-hand side, the second factor

$$\nabla_y \cdot ((y-x)/|y-x|^3) \tag{3.15}$$

is the divergence of the well known ''inverse square central field.'' Using spherical coordinates centered at x, this can be written as

$$\nabla \cdot \hat{r}/r^2 = (1/r^2)(\partial/\partial r)(r^2(1/r^2)) = 0, \tag{3.16}$$

away from the origin.

But the integral of $\nabla \cdot \hat{r}/r^2$ over any ball centered at the origin, when converted to a surface integral via the divergence theorem, is clearly 4π,

$$\int_{\mathrm{ball}} \nabla \cdot \hat{r}/r^2 \ d(\mathrm{vol}) = \int_{\mathrm{sphere}} (\hat{r}/r^2) \cdot n \ d(\mathrm{area}) = 4\pi. \tag{3.17}$$

Thus,

$$\nabla \cdot \hat{r}/r^2 = 4\pi\delta^3(r), \tag{3.18}$$

where $\delta^3(r)$ is the three-dimensional delta function; equivalently,

$$\nabla_y \cdot ((y-x)/|y-x|^3) = 4\pi \delta^3(y-x). \tag{3.19}$$

Hence,

$$(1/4\pi)\int_\Omega V(x)\nabla_y \cdot((y-x)/|y-x|^3) \ \mathrm{d(vol}_x) = (1/4\pi)\int_\Omega V(x)4\pi\delta^3(y-x) \ \mathrm{d(vol}_x)$$

$$= V(y) \ \text{in} \ \Omega/0 \ \text{in} \ \Omega'. \tag{3.20}$$

Thus far, we have proved that

$$\nabla_y \times \mathrm{BS}(V)(y) = \{V(y)/0\} - (1/4\pi)\int_\Omega (V(x)\cdot\nabla_y)((y-x)/|y-x|^3) \ \mathrm{d(vol}_x). \tag{3.21}$$

Now we focus on the second term on the right-hand side and must show that

$$-(1/4\pi)\int_\Omega (V(x)\cdot\nabla_y)((y-x)/|y-x|^3) \ \mathrm{d(vol}_x)$$

$$= (1/4\pi)\nabla_y \int_\Omega (\nabla_x \cdot V(x))/|y-x| \ \mathrm{d(vol}_x)$$

$$- (1/4\pi)\nabla_y \int_{\partial\Omega} V(x)\cdot n/|y-x| \ \mathrm{d(area}_x). \tag{3.22}$$

We begin by writing each of the three terms in (3.22) in the form

$$\pm(1/4\pi)\nabla_y \int_\Omega (\text{something}) \ \mathrm{d(vol}_x). \tag{3.23}$$

Starting with the left-hand side of (3.22), we claim that its integrand can be rewritten as

$$(V(x)\cdot\nabla_y)((y-x)/|y-x|^3) = \nabla_y(V(x)\cdot(y-x)/|y-x|^3). \tag{3.24}$$

To see this, we need the formula from vector calculus,

$$\nabla(V\cdot W) = V\times(\nabla\times W) + W\times(\nabla\times V) + (V\cdot\nabla)W + (W\cdot\nabla)V. \tag{3.25}$$

We use this with $\nabla = \nabla_y$, $V = V(x)$, and $W = (y-x)/|y-x|^3$. Three of the four terms on the right-hand side of (3.25) will then be zero; the first is zero because $\nabla_y \times W = 0$; the second is zero because $\nabla_y \times V(x) = 0$; the fourth is zero because $(W\cdot\nabla_y)V(x) = 0$. Thus $\nabla_y(V\cdot W) = (V\cdot\nabla_y)W$, which is exactly our claim.

The first term on the right-hand side of (3.22) is already in the desired form.

The second term on the right-hand side of (3.22) can be rewritten as

$$\int_{\partial\Omega} V(x)\cdot n/|y-x| \ \mathrm{d(area}_x) = \int_\Omega \nabla_x\cdot(V(x)/|y-x|) \ \mathrm{d(vol}_x), \tag{3.26}$$

thanks to the divergence theorem.

Now that all the terms in (3.22) have been rewritten in the desired form, we claim that the integrands on both sides are equal, namely, that

$$-V(x)\cdot(y-x)/|y-x|^3 = (\nabla_x\cdot V(x))/|y-x| - \nabla_x\cdot(V(x)/|y-x|). \tag{3.27}$$

This is an immediate consequence of the formula

$$\nabla \cdot (fA) = (\nabla f) \cdot A + f(\nabla \cdot A), \tag{3.28}$$

and the proof of Proposition 1 is complete.

C. Examples

We give three examples to illustrate Proposition 1, each in ''bare bones'' format, and invite the interested reader to carry out the supporting calculations.

Example 1: In this example, we start with the vector field

$$V = \partial/\partial z = \hat{z} \tag{3.29}$$

on the ball Ω of radius a centered at the origin. Note that $V \epsilon \, CG(\Omega)$.

Switching to spherical coordinates (r, θ, φ), a straightforward computation yields

$$BS(V) = (a^3/3)(\sin \theta)/r^2 \hat{\varphi} \ \ \text{for} \ \ r \geq a$$

$$= (1/3) r \sin \theta \, \hat{\varphi} \ \ \text{for} \ \ r \leq a. \tag{3.30}$$

Note that inside the ball, $BS(V)$ coincides with the velocity field of a body rotating with constant angular velocity about the z-axis.

Next we compute $\nabla \times BS(V)$,

$$\nabla \times BS(V) = (a^3/3)\{(2 \cos \theta/r^3)\hat{r} + (\sin \theta/r^3)\hat{\theta}\} \ \ \text{for} \ \ r \geq a, \tag{3.31}$$

which is a standard dipole field, while

$$\nabla \times BS(V) = (2/3)\{(\cos \theta)\hat{r} - (\sin \theta)\hat{\theta}\} = (2/3)V \ \ \text{for} \ \ r \leq a. \tag{3.32}$$

We invite the reader to check Proposition 1, equivalently the Maxwell equation (3.10), inside the domain Ω by directly computing that $E'_\sigma = (-1/3)V$ there.

Example 2 (see Example 4 of Sec. II C): In this example, we start with the function $f = 1/r$ on the domain Ω between the spheres of radii 1 and 2 centered at the origin, and then consider the vector field

$$V = \nabla f = -\hat{r}/r^2 \tag{3.33}$$

on this domain. Note that the function f is harmonic, and is constant on each component of $\partial \Omega$. Therefore V lies in the subspace $HG(\Omega)$ of harmonic gradients inside $VF(\Omega)$. Borrowing once again from the future proof of Theorem B, we note that V lies in the kernel of the Biot–Savart operator.

We invite the reader to confirm Maxwell's equation (3.10) by checking that

$$E_\sigma = \hat{r}/r^2 \ \ \text{inside} \ \ \Omega,$$

$$= 0 \ \ \ \ \text{outside} \ \ \Omega. \tag{3.34}$$

Example 3 (see Example 5 of Sec. II C): In this example, we start with the function $f(x,y,z) = x^2 + y^2 + z^2 - 1 = r^2 - 1$ on the unit ball Ω centered at the origin, and then consider the vector field

$$V = \nabla f = 2r\hat{r} \tag{3.35}$$

on this ball. Note that V lies in the subspace $GG(\Omega)$ of grounded gradients inside $VF(\Omega)$, and is therefore (borrowing from the future proof of Theorem B) in the kernel of the Biot–Savart operator BS.

With this in mind, we invite the reader to confirm Maxwell's equation (3.10) in this case by computing that

$$E_\rho = -2r\hat{r} \quad \text{inside} \ \ \Omega$$
$$= -2\hat{r}/r^2 \ \ \text{outside} \ \ \Omega, \tag{3.36}$$

and that

$$E_\sigma = 0 \qquad \text{inside} \ \ \Omega$$
$$= 2\hat{r}/r^2 \ \ \text{outside} \ \ \Omega. \tag{3.37}$$

IV. PROOF OF THEOREM A

Recall the statement:

Theorem A: *The equation $\nabla \times BS(V) = V$ holds in Ω if and only if V is divergence-free and tangent to the boundary of Ω.*

The condition that V be divergence-free and tangent to the boundary of Ω can also be written as $V \epsilon K(\Omega) = FK \oplus HK$, the subspace of fluid knots. For the same effort, we will also get

Addendum to Theorem A: *The equation $\nabla \times BS(V) = 0$ holds in the closure Ω' of $R^3 - \Omega$ if and only if $V \epsilon FK \oplus HK \oplus HG \oplus GG$.*

This condition on V is equivalent to V being orthogonal to the subspace CG of curly gradients in VF(Ω). Then we will prove.

Corollary to Theorem A: *The vector potential $A(V)$ is divergence-free if and only if V is divergence-free and tangent to the boundary of Ω.*

A. Proof of Theorem A

Half of Theorem A has already appeared as item (7) in our list of Standard Information, and was proved in Sec. III B, namely, if $V \epsilon K(\Omega) = FK \oplus HK$, then $\nabla \times BS(V) = V$ in Ω.

By contrast, if $V \epsilon HG \oplus GG$, then it would be impossible for $\nabla \times BS(V)$ to equal V in Ω unless $V = 0$, since we know from the Hodge Decomposition Theorem that the image of curl is $FK \oplus HK \oplus CG$.

It remains to show that if V is in CG, then $\nabla \times BS(V)$ can never equal V in Ω unless $V = 0$. The proof will be based on the Maxwell equation,

$$\nabla_y \times BS(V)(y) = \{V(y) \ \text{in} \ \Omega / 0 \ \text{in} \ \Omega'\} - (1/4\pi)\nabla_y \int_{x \epsilon \partial\Omega} V(x) \cdot n(x)/|x-y| \ \ d(\text{area}_x). \tag{4.1}$$

Following our discussion in Sec. III B, we can write the second term on the right-hand side of this equation as

$$E'_\sigma(y) = -(1/4\pi)\nabla_y \int_{x \epsilon \partial\Omega} V(x) \cdot n(x)/|x-y| \ \ d(\text{area}_x). \tag{4.2}$$

Although E'_σ is the time rate of change of the electrostatic field E_σ, it is also the same as the electrostatic field due to a charge density $\sigma(x) = V(x) \cdot n(x)$ along $\partial\Omega$, and so we can treat it as though it were an electrostatic field.

We write

$$E'_\sigma(y) = -\nabla_y \psi(y), \tag{4.3}$$

where

$$\psi(y) = (1/4\pi) \int_{x \in \partial\Omega} V(x) \cdot n(x)/|x-y| \; d(\text{area}_x).$$ (4.4)

Although we have in general been writing our gradient fields with a plus sign, as in the equation $V = \nabla\varphi$, we write electrostatic fields with a minus sign, $E_\sigma' = -\nabla\psi$, to follow standard convention.

While the electrostatic potential function ψ for a surface charge distribution σ is continuous, the electrostatic field E_σ' will in general have a jump discontinuity as we cross the surface. Nevertheless, we have $\nabla \cdot E_\sigma' = 0$ in Ω and $\nabla \cdot E_\sigma' = 0$ in Ω'.

We claim that if V is a nonzero vector field in CG, then E_σ' cannot be identically zero in Ω.

Recall the definition of the subspace CG of curly gradients. A smooth vector field V defined on Ω is in CG if and only if $V = \nabla\varphi$, where φ is a harmonic function on Ω, and where the flux of V through each component of $\partial\Omega$ is zero. That is, for each component $\partial\Omega_i$ of $\partial\Omega$, we have

$$\int_{\partial\Omega_i} V(x) \cdot n(x) \; d(\text{area}_x) = \int_{\partial\Omega_i} \sigma(x) \; d(\text{area}_x) = 0.$$ (4.5)

In other words, the total charge on each component of $\partial\Omega$ is zero.

Suppose now that $E_\sigma' = 0$ in Ω. We must show that $V = 0$.

First we will show that $E_\sigma' = 0$ in Ω', the closure of $R^3 - \Omega$.

The hypothesis that $E_\sigma' = 0$ inside Ω tells us that ψ must be constant on each component $\partial\Omega_i$ of $\partial\Omega' = \partial\Omega$.

Now we consider the field $\psi E_\sigma'$ in Ω', and compute its divergence (a standard trick in electrostatics),

$$\nabla \cdot (\psi E_\sigma') = (\nabla\psi) \cdot E_\sigma' + \psi(\nabla \cdot E_\sigma') = -E_\sigma' \cdot E_\sigma' = -|E_\sigma'|^2.$$ (4.6)

Hence,

$$\int_{\Omega'} |E_\sigma'|^2 \; d(\text{vol}) = -\int_{\Omega'} \nabla \cdot (\psi E_\sigma') \; d(\text{vol}) = -\int_{\partial\Omega'} \psi E_\sigma' \cdot n' d(\text{area}),$$ (4.7)

where n' is the unit outward-pointing normal vector to Ω', so that $n' = -n$.

Using the divergence theorem in Ω' requires a comment, since one of its components is unbounded. That unbounded component should really be approximated by a bounded domain with one boundary component out near infinity. The flux of $\psi E_\sigma'$ through this boundary component goes to zero as it recedes towards infinity, because the area grows like r^2, while the field E_σ' decays like $1/r^2$ and the potential ψ decays like $1/r$.

With that said, we continue,

$$\int_{\Omega'} |E_\sigma'|^2 \; d(\text{vol}) = -\int_{\partial\Omega'} \psi E_\sigma' \cdot n' d(\text{area}) = -\sum_i \psi_i \int_{\partial\Omega_i} E_\sigma' \cdot n' d(\text{area}),$$ (4.8)

since ψ is constant, say with value ψ_i, on each component $\partial\Omega_i$ of the boundary.

Now, by Gauss' Law,

$$\int_{\partial\Omega_i} E_\sigma' \cdot n' d(\text{area}) = \pm \text{total charge ''inside''} \; \partial\Omega_i$$

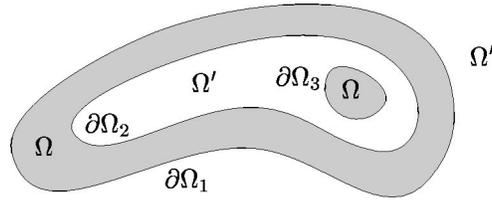$$= \pm \sum_{\text{some } j} \int_{\partial\Omega_j} \sigma(x) \; d(\text{area}_x) = 0,$$ (4.9)

FIG. 1. Components of the domain Ω and of its complement, Ω'.

because the total charge on *each* component $\partial\Omega_j$ of $\partial\Omega$ is zero (see Fig. 1).

Thus, $\int_{\Omega'}|E'_\sigma|^2 d(\text{vol})=0$, and hence $E'_\sigma\equiv 0$ in Ω'.

Now we have $E'_\sigma\equiv 0$ in Ω and also in Ω'. Then Gauss's Law, applied to the typical ''pill box'' neighborhood of a point on $\partial\Omega$, implies that the surface charge distribution σ is identically zero (see Fig. 2).

Since $\sigma(x)=V(x)\cdot n(x)$, this imples that V is tangent to the boundary of Ω, and hence $V\in K(\Omega)$. But $K(\Omega)\cap CG=0$, so $V=0$.

This completes the proof of Theorem A.

B. Proof of Addendum to Theorem A

We know that if V lies in $K(\Omega)=FK\oplus HK$, then $\nabla\times BS(V)=0$ in Ω', according to item (7) in our list of Standard Information from Sec. III A.

Borrowing from the future, we will see in the proof of Theorem B that if $V\epsilon HG\oplus GG$, then $BS(V)=0$ throughout 3-space, so that surely $\nabla\times BS(V)=0$ in Ω'.

This gives us half of the Addendum to Theorem A.

It remains to show that if V is in CG, then $\nabla\times BS(V)$ cannot be zero in Ω' unless $V=0$ in Ω.

The proof of this is based on the Maxwell equation (4.1), as was the proof of Theorem A; it is a copy of the argument given there, with the roles of Ω and Ω' reversed, so we omit further details.

C. Proof of Corollary to Theorem A

If V is divergence-free and tangent to the boundary of Ω, then we already know from item (5) in the list of Standard Information that the vector potential $A(V)$ is divergence-free.

Recall, also from that list, items

(3) $\Delta A(V)=-V$, and

(4) $\nabla\times A(V)=BS(V)$ for all $V\epsilon$ VF(Ω).

Now take the second derivative formula,

$$\nabla\times(\nabla\times W)=\nabla(\nabla\cdot W)-\Delta W \tag{4.10}$$

for any vector field W, and rewrite it with A(V) in place of W,
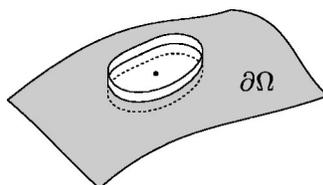

FIG. 2. A typical ''pill box'' neighborhood of a point on $\partial\Omega$.

$$\nabla \times (\nabla \times A(V)) = \nabla(\nabla \cdot A(V)) - \Delta A(V). \tag{4.11}$$

Using items (3) and (4) above, substitute BS(V) for $\nabla \times A(V)$ on the left-hand side, and V for $-\Delta A(V)$ on the right-hand side, to get

$$\nabla \times BS(V) = \nabla(\nabla \cdot A(V)) + V. \tag{4.12}$$

If A(V) is divergence-free, then we get

$$\nabla \times BS(V) = V \quad \text{inside } \Omega, \tag{4.13}$$

which by Theorem A implies that $V \epsilon$ K(Ω).

We conclude that A(V) is divergence-free if and only if $V \epsilon$ K(Ω), which is exactly the assertion of the Corollary.

V. PROOF OF THEOREM B

A. Proof of Theorem B, easy direction

Recall the statement:

Theorem B: *The kernel of the Biot–Savart operator is precisely the space of gradient vector fields which are orthogonal to the boundary of* Ω.

The easy direction is to assume that V is a gradient vector field which is orthogonal to the boundary of Ω (equivalently, that $V \epsilon$ HG$\oplus$GG), and then conclude that BS(V)=0. We will do that here, and will actually show that BS(V)=0 throughout all of 3-space, rather than just in Ω.

We begin with the following lemma, which is stated without proof on p. 60 of Griffiths.[46]

Lemma 1: *Let V be a smooth vector field on the domain* Ω, *and let n denote the outward pointing unit normal vector field to* $\partial\Omega$. *Then,*

$$\int_\Omega \nabla \times V \, d(\text{vol}) = -\int_{\partial\Omega} V \times n \, d(\text{area}).$$

Proof: Start with the Divergence Theorem,

$$\int_\Omega \nabla \cdot V d(\text{vol}) = \int_{\partial\Omega} V \cdot n \; d(\text{area}).$$

Then replace V by $V \times C$, where C is any constant vector,

$$\int_\Omega \nabla \cdot (V \times C) \; d(\text{vol}) = \int_{\partial\Omega} (V \times C) \cdot n \; d(\text{area}).$$

Writing $\nabla \cdot (V \times C) = (\nabla \times V) \cdot C$ and moving C outside the integral, the left-hand side becomes

$$C \cdot \int_\Omega \nabla \times V \; d(\text{vol}).$$

Writing $(V \times C) \cdot n = -(V \times n) \cdot C$ and again moving C outside the integral, the right-hand side becomes

$$-C \cdot \int_{\partial\Omega} V \times n \; d(\text{area}).$$

Since the left- and right-hand sides are equal for all C, we must have

$$\int_\Omega \nabla \times V \ \mathrm{d(vol)} = - \int_{\partial\Omega} V \times n \ \mathrm{d(area)},$$

proving the lemma.

Suppose now that $V = \nabla\varphi$ is a gradient vector field on Ω which is orthogonal to the boundary, which means that φ is constant on each component $\partial\Omega_i$ of $\partial\Omega$. We must show that $\mathrm{BS}(V) = 0$.

Begin with the formula for the Biot–Savart operator,

$$\mathrm{BS}(V)(y) = (1/4\pi)\int_\Omega V(x) \times (y-x)/|y-x|^3 \ \mathrm{d(vol}_x). \tag{5.1}$$

Fix y, and let $W = (y-x)/|y-x|^3$. Then,

$$\mathrm{BS}(V) = (1/4\pi)\int_\Omega (\nabla\varphi) \times W \ \mathrm{d(vol)}. \tag{5.2}$$

Now consider the vector field φW on Ω and take its curl,

$$\nabla \times (\varphi W) = (\nabla\varphi) \times W + \varphi(\nabla \times W) = (\nabla\varphi) \times W, \tag{5.3}$$

since $\nabla \times W = 0$. Thus

$$\mathrm{BS}(V) = (1/4\pi)\int_\Omega \nabla \times (\varphi W) \ \mathrm{d(vol)}. \tag{5.4}$$

We would like to use the preceding lemma to replace the right-hand side of this formula by the expression

$$-(1/4\pi)\int_{\partial\Omega} (\varphi W) \times n \ \mathrm{d(area)}. \tag{5.5}$$

But the vector field φW does not quite fit the hypothesis of the lemma, since it has an isolated singularity at the point y (which we can assume is in the interior of Ω). However, this singularity is ''radial;'' if we surround it by a small sphere, the vector field φW will be orthogonal to the sphere, and so the integral $\int (\varphi W) \times n \ \mathrm{d(area)}$ over this small sphere will be zero. It follows immediately that the lemma can be applied in this case, in spite of the singularity.

We do so, and continue

$$\mathrm{BS}(V) = -(1/4\pi)\int_{\partial\Omega} (\varphi W) \times n \ \mathrm{d(area)}$$

$$= -(1/4\pi)\sum_i \varphi_i \int_{\partial\Omega_i} W \times n \ \mathrm{d(area)}, \tag{5.6}$$

where φ_i is the constant value of φ on $\partial\Omega_i$.

Now we claim that, for each i,

$$\int_{\partial\Omega_i} W \times n \ \mathrm{d(area)} = 0. \tag{5.7}$$

To see this, let Ω_i be the compact domain in 3-space bounded by $\partial\Omega_i$. Then, using the lemma once again,

$$\int_{\partial\Omega_i} W\times n \ \ \mathrm{d(area)}=\pm\int_{\Omega_i}\nabla\times W \ \ \mathrm{d(vol)} \tag{5.8}$$

with the $+$ or $-$ sign chosen according as n points into or out of Ω_i. In any case, $\nabla\times W=0$, so the integral vanishes.

Thus $\mathrm{BS}(V)=0$ throughout 3-space.

B. Proof of Theorem B, harder direction

The heart of the argument is the following energy estimate.

Proposition 2: *Let Ω be a compact domain with smooth boundary in 3-space, and V a smooth divergence-free vector field defined in Ω. Let E'_σ be the electrostatic field due to the charge distribution $\sigma(x)=V(x)\cdot n(x)$ along $\partial\Omega$. Then,*

$$\int_{3\text{-space}}|E'_\sigma|^2 \ \ \mathrm{d(vol)}\leqslant\int_\Omega|V|^2 \ \ \mathrm{d(vol)}.$$

That is, the energy of the electrostatic field E'_σ throughout all of 3-space is bounded from above by the energy of the original field V in Ω.

When V is not required to be divergence-free, the energy of the field E'_σ can be made arbitrarily large, while keeping the energy of V itself as small as desired: make $V(x)\cdot n(x)$ large along $\partial\Omega$, and then quickly taper V off to zero throughout most of Ω.

Proof of Proposition 2: Given a divergence-free vector field V, we can subtract from V its orthogonal projection into the space $\mathrm{K}(\Omega)=\mathrm{FK}\oplus\mathrm{HK}$ of fluid knots. This will leave the corresponding electrostatic field E'_σ unchanged, while at worst decreasing the energy in V.

So in proving the proposition, there is no loss in generality in assuming that V is already orthogonal to this subspace, and hence a gradient vector field...as well as being divergence-free. Thus we can write

$$V=\nabla\varphi \ \ \text{with} \ \ \Delta\varphi=0. \tag{5.9}$$

Likewise,

$$E'_\sigma(y)=-\nabla_y\psi(y), \tag{5.10}$$

where

$$\psi(y)=(1/4\pi)\int_{x\epsilon\partial\Omega} V(x)\cdot n(x)/|x-y| \ \ \mathrm{d(area}_x). \tag{5.11}$$

Lemma 2: $\displaystyle\int_{3\text{-space}}|E'_\sigma|^2\mathrm{d(vol)}=\int_{\partial\Omega}\psi\partial\varphi/\partial n\,\mathrm{d(area)}.$

Proof of Lemma 2: This is a standard result in electrostatics; see Griffiths[46] pp. 94–95. For convenience, we give the argument here.

Since the surface charge distribution σ along $\partial\Omega$ is given by

$$\sigma(x)=V(x)\cdot n(x)=(\nabla\varphi(x))\cdot n(x)=(\partial\varphi/\partial n)(x), \tag{5.12}$$

we can rewrite the equation to be proved as

$$\int_{3\text{-space}}|E'_\sigma|^2 \ \ \mathrm{d(vol)}=\int_{\partial\Omega}\psi\sigma \ \ \mathrm{d(area)}. \tag{5.13}$$

This more clearly displays the relation of the integrand on the right-hand side to the field E_σ'; the function σ is the surface charge distribution along $\partial\Omega$ which gives rise to the field E_σ', while the function ψ is the electrostatic potential for E_σ', that is, $E_\sigma' = -\nabla\psi$.

The proof is a little easier to express if we replace the surface charge distribution σ by a volume charge distribution ρ in a small neighborhood $N(\partial\Omega)$ of $\partial\Omega$, and let $E_\rho' = -\nabla\psi$ be the resulting electrostatic field, because in this situation we can write $\nabla\cdot E_\rho = \rho$.

With this understanding, we must show that

$$\int_{3\text{-space}} |E_\rho'|^2 \ \mathrm{d(vol)} = \int_{N(\partial\Omega)} \psi\rho \ \mathrm{d(vol)}. \tag{5.14}$$

To prove this, rewrite the integral on the right-hand side as

$$\int_{N(\partial\Omega)} \psi(\nabla\cdot E_\rho') \ \mathrm{d(vol)}. \tag{5.15}$$

Next,

$$\nabla\cdot(\psi E_\rho') = (\nabla\psi)\cdot E_\rho' + \psi(\nabla\cdot E_\rho') = -|E_\rho'|^2 + \psi(\nabla\cdot E_\rho'). \tag{5.16}$$

Hence

$$\int_{N(\partial\Omega)} \psi\rho \ \mathrm{d(vol)} = \int_{N(\partial\Omega)} \psi(\nabla\cdot E_\rho') \ \mathrm{d(vol)}$$

$$= \int_{N(\partial\Omega)} \nabla\cdot(\psi E_\rho') \ \mathrm{d(vol)} + \int_{N(\partial\Omega)} |E_\rho'|^2 \ \mathrm{d(vol)}. \tag{5.17}$$

If, in the integral on the left-hand side above, we replace the neighborhood $N(\partial\Omega)$ by any larger domain, call it Ω^*, the value of the integral will not change because $\rho = 0$ outside $N(\partial\Omega)$. And the equation above will still hold if we replace $N(\partial\Omega)$ by Ω^* in each of the three integrals,

$$\int_{\Omega^*} \psi\rho \ \mathrm{d(vol)} = \int_{\Omega^*} \nabla\cdot(\psi E_\rho') \ \mathrm{d(vol)} + \int_{\Omega^*} |E_\rho'|^2 \ \mathrm{d(vol)}. \tag{5.18}$$

Apply the divergence theorem to the first integral on the right-hand side, so that we now have

$$\int_{\Omega^*} \psi\rho \ \mathrm{d(vol)} = \int_{\partial\Omega^*} (\psi E_\rho')\cdot n \ \mathrm{d(area)} + \int_{\Omega^*} |E_\rho'|^2 \ \mathrm{d(vol)}. \tag{5.19}$$

Visualize the domain Ω^* growing larger and larger, with its boundary receding towards infinity. Then ψ decays like $1/r$, while E_ρ' decays like $1/r^2$ and the area of $\partial\Omega^*$ grows like r^2. Thus the value of the first integral on the right-hand side decays like $1/r$, and so goes to zero in the limit. Hence

$$\int_{N(\partial\Omega)} \psi\rho \ \mathrm{d(vol)} = \int_{3\text{-space}} |E_\rho'|^2 \ \mathrm{d(vol)}, \tag{5.20}$$

the desired result for volume charge distributions.

If we compress the neighborhood $N(\partial\Omega)$ towards the surface $\partial\Omega$, the above result for volume charge distributions will tend to the corresponding result for surface charge distributions,

$$\int_{\partial\Omega} \psi\sigma \ \mathrm{d(area)} = \int_{3\text{-space}} |E_\sigma'|^2 \ \mathrm{d(vol)}, \tag{5.21}$$

and the lemma is proved.

Completion of the proof of Proposition 2: Now we recall Green's first identity.
Let $A = \psi \nabla \varphi$. Then

$$\nabla \cdot A = \nabla \cdot (\psi \nabla \varphi) = \nabla \psi \cdot \nabla \varphi + \psi \Delta \varphi = \nabla \psi \cdot \nabla \varphi, \tag{5.22}$$

since $\Delta \varphi = 0$.

Thus,

$$\int_\Omega - E'_\sigma \cdot V \ \mathrm{d(vol)} = \int_\Omega \nabla \psi \cdot \nabla \varphi \ \mathrm{d(vol)} = \int_\Omega \nabla \cdot A \ \mathrm{d(vol)} = \int_{\partial \Omega} A \cdot n \ \mathrm{d(area)}$$

$$= \int_{\partial \Omega} \psi \nabla \varphi \cdot n \ \mathrm{d(area)} = \int_{\partial \Omega} \psi \partial \varphi / \partial n \ \mathrm{d(area)} = \int_{3\text{-space}} |E'_\sigma|^2 \ \mathrm{d(vol)}, \quad (5.23)$$

by the lemma.

Hence,

$$\int_{3\text{-space}} |E'_\sigma|^2 \ \mathrm{d(vol)} = \int_\Omega - E'_\sigma \cdot V \mathrm{d(vol)}$$

$$\leqslant \left(\int_\Omega |E'_\sigma|^2 \ \mathrm{d(vol)} \right)^{1/2} \left(\int_\Omega |V|^2 \ \mathrm{d(vol)} \right)^{1/2}$$

$$\leqslant \left(\int_{3\text{-space}} |E'_\sigma|^2 \ \mathrm{d(vol)} \right)^{1/2} \left(\int_\Omega |V|^2 \ \mathrm{d(vol)} \right)^{1/2}, \tag{5.24}$$

and therefore

$$\int_{3\text{-space}} |E'_\sigma|^2 \ \mathrm{d(vol)} \leqslant \int_\Omega |V|^2 \ \mathrm{d(vol)}, \tag{5.25}$$

as claimed, finishing the proof of Proposition 2.

Completion of the proof of Theorem B: In the previous section, we showed that HG⊕GG, the space of gradient vector fields which are orthogonal to the boundary of Ω, lies within the kernel of the Biot–Savart operator BS:VF$(\Omega) \to$ VF(Ω).

Now we must show that there is nothing else in the kernel.

We will do this by assuming that V is orthogonal to GG (equivalently, is divergence-free) and that BS$(V) = 0$, and will show that V must lie in HG.

First we observe that, under these assumptions, V must be a gradient vector field.

To see this, consider the Maxwell equation in Ω,

$$\nabla_y \times \mathrm{BS}(V)(y) = V(y) - (1/4\pi) \nabla_y \int_{x \epsilon \partial \Omega} V(x) \cdot n(x) / |x - y| \ \mathrm{d(area}_x), \tag{5.26}$$

written in the form appropriate for any divergence-free vector field V.

If V had a nonzero component in the subspace FK⊕HK of fluid knots, then that component would persist when we computed $\nabla \times \mathrm{BS}(V)$, since the Maxwell equation tells us that $\nabla \times \mathrm{BS}(V)$ differs from V by a gradient vector field. It follows that no such V could possibly be in the kernel of BS.

So we can assume that V is a gradient vector field, and write $V = \nabla \varphi$. Since V is orthogonal to GG, the function φ must be harmonic. To show that V lies in HG, we must show that the function φ is constant on each component of $\partial \Omega$.

To start on this, note that the second term on the right-hand side in the Maxwell equation above is the electrostatic field $E'_\sigma(y)$, and write that equation more succinctly as

$$\nabla \times \mathrm{BS}(V) = V + E'_\sigma. \tag{5.27}$$

Now if $\mathrm{BS}(V) = 0$, then $E'_\sigma = -V$ in Ω.

It follows that E'_σ must be identically zero outside Ω because, by Proposition 2, it simply has no more energy.

This, in turn, implies that the electrostatic potential function ψ for the field E'_σ must be constant on each component of $\partial\Omega$.

But the three equations,

$$E'_\sigma = -\nabla\psi \quad \text{(everywhere)}, \tag{5.28}$$

$$V = \nabla\varphi \quad \text{(inside } \Omega), \tag{5.29}$$

$$E'_\sigma = -V \quad \text{(inside } \Omega), \tag{5.30}$$

tell us that

$$\nabla\varphi = \nabla\psi \quad \text{(inside } \Omega), \tag{5.31}$$

and hence that

$$\varphi = \psi + \text{ some constant} \tag{5.32}$$

on each component of Ω, where the constant may depend on the component.

Thus φ inherits from ψ the property of being constant on each component of $\partial\Omega$, and hence $V = \nabla\varphi$ must lie in HG, the desired conclusion.

This completes the proof of Theorem B.

In fact, we have actually proved a bit more.

Theorem B′: *The kernel of $\nabla \times BS$, the composition of the curl and Biot–Savart operators, is also the space of gradient vector fields which are orthogonal to the boundary of Ω.*

This follows, with no further argument, because the only way we used the hypothesis that $\mathrm{BS}(V) = 0$ in this section was to set $\nabla \times \mathrm{BS}(V) = 0$ on the left-hand side of the Maxwell equation (5.27).

VI. PROOF OF THEOREM C

A. Statement and proof of Theorem C

Recall the statement:

Theorem C: *The image of the Biot–Savart operator is a proper subspace of the image of curl, whose orthogonal projection into the subspace of "fluxless knots" is one-to-one.*

This will follow immediately from Theorems B and B′ and (borrowing from the future) from Theorem D.

Proof:

Keep in mind the Hodge decomposition,

$$\mathrm{VF}(\Omega) = \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG}. \tag{6.1}$$

We know from Theorem B that the kernel of the Biot–Savart operator BS is the subspace $\mathrm{HG} \oplus \mathrm{GG}$ of $\mathrm{VF}(\Omega)$.

We know from Theorem D that this operator is self-adjoint.

It follows that the image of BS lies within the orthogonal complement of its kernel, that is, within the subspace $\mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG}$, which is precisely the image of curl.

Alternatively, the formula $\nabla \times A(V) = BS(V)$, which appeared as item (4) on our list of Standard Information in Sec. III A, also tells us that the image of BS lies within the image of curl.

Now it follows from Theorems B and B$'$ together that

$$\text{Image(BS)} \cap \text{Ker(curl)} = \{0\}, \tag{6.2}$$

and since, by the Hodge Decomposition Theorem, the kernel of curl is HK$\oplus$CG$\oplus$HG$\oplus$GG, the orthogonal projection of the image of BS into FK must be one-to-one.

From this it also follows that the image of the BS is a proper subspace of the image of curl.

This completes the proof of Theorem C.

B. Impossible magnetic fields

We are looking for smooth vector fields U on a compact, smoothly bounded domain Ω in 3-space, for which it is impossible to find a smooth vector field V on Ω satisfying the equation $U = BS(V)$. We will call such a field U an impossible magnetic field.

Of course, Eq. (6.2) tells us that any nonzero vector field U in HK$\oplus$CG$\oplus$HG$\oplus$GG is an impossible magnetic field.

But here is a more interesting example.

Consider the velocity vector field U of a ''speeding bullet,'' as pictured below (see Fig. 3).

We visualize the unit ball Ω in 3-space as a lead bullet sitting in a cartridge which has been shot directly upwards from a rifled barrel, so that it spins as it moves forward. In cylindrical coordinates r, φ, z, the velocity vector field U is given by

$$U = r\hat{\varphi} + \hat{z}. \tag{6.3}$$

Note that the first summand $r\hat{\varphi}$ lies in FK, while the second summand $\hat{z}$ lies in CG.

Now look back to Example 1 in Sec. III C. There we started with the vector field $V = \hat{z}$ on the unit ball Ω and computed its magnetic field within the ball,
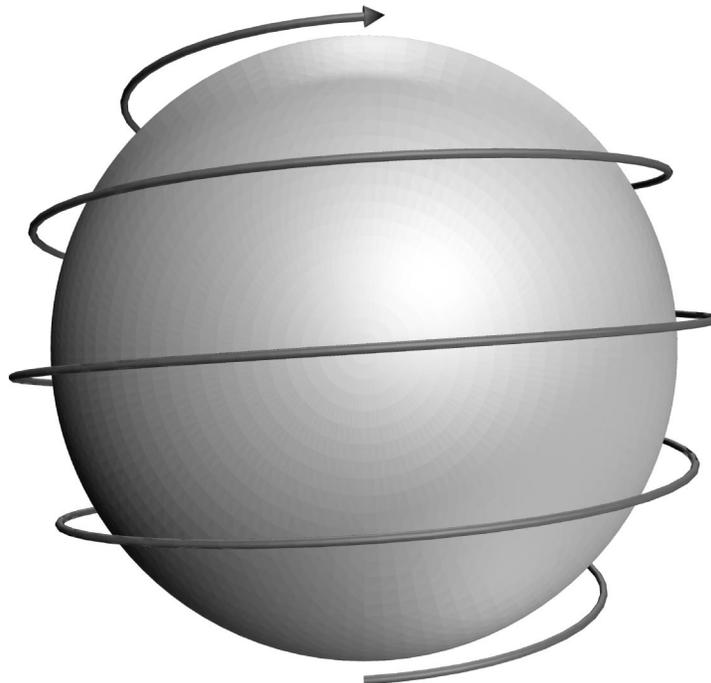

FIG. 3. An impossible magnetic field on the unit ball Ω.

$$\mathrm{BS}(V) = (1/3)r\sin\theta\,\hat{\varphi} \quad \text{(spherical coordinates)}$$

$$= (1/3)r\hat{\varphi} \quad \text{(cylindrical coordinates).} \tag{6.4}$$

So of course (settling back to cylindrical coordinates),

$$\mathrm{BS}(3V) = r\hat{\varphi}. \tag{6.5}$$

But this magnetic field on the unit ball ''poisons'' U as a candidate magnetic field, since U and $\mathrm{BS}(3V)$ have the same orthogonal projection into the space FK of fluxless knots. By Theorem C, the vector field U cannot possibly be the Biot–Savart transform of any smooth vector field on Ω.

VII. PROOF OF THEOREM D

It will be convenient to divide the statement and proof of Theorem D into three pieces, as follows:

(1) The Biot–Savart operator $\mathrm{BS}{:}\mathrm{VF}(\Omega)\to\mathrm{VF}(\Omega)$ is bounded, and hence extends to a bounded operator on the L^2 completion,

$$\mathrm{BS}{:}\overline{\mathrm{VF}}(\Omega)\to\overline{\mathrm{VF}}(\Omega);$$

(2) The operator $\mathrm{BS}{:}\overline{\mathrm{VF}}(\Omega)\to\overline{\mathrm{VF}}(\Omega)$ is compact, that is, it takes the unit ball in $\overline{\mathrm{VF}}(\Omega)$ to a set with compact closure in $\overline{\mathrm{VF}}(\Omega)$;
(3) The operator $\mathrm{BS}{:}\overline{\mathrm{VF}}(\Omega)\to\overline{\mathrm{VF}}(\Omega)$ is self-adjoint with respect to the L^2 inner product, that is, $\langle V_1,\mathrm{BS}(V_2)\rangle=\langle\mathrm{BS}(V_1),V_2\rangle$, for all vector fields V_1 and V_2 in $\overline{\mathrm{VF}}(\Omega)$.

A. A useful lemma

The proof that the Biot–Savart operator is bounded, as asserted in (1) above, will follow along the lines of the usual Young's inequality proof that convolution operators are bounded; see Folland,[47] p. 9, or Zimmer,[48] Proposition B.3 on p. 10. We extract this proof as a lemma, so that we can use it again in the proof of part (2).

Lemma 3: *Let $\phi(x)$ be a scalar-valued function with the property that*

$$N_\Omega(\phi)=\max_y\int_\Omega|\phi(y-x)|\ \mathrm{d}(\mathrm{vol}_x)$$

is finite, where the maximum is over all points $y\,\epsilon R^3$. Then the operator $T_\phi{:}\mathrm{VF}(\Omega)\to\mathrm{VF}(\Omega)$ defined by

$$T_\phi(V)(y)=\int_\Omega V(x)\times\phi(y-x)\,\frac{y-x}{|y-x|}\,\mathrm{d}(\mathrm{vol}_x)$$

is a bounded map with respect to the L^2 norm, and furthermore,

$$|T_\phi(V)|\leqslant N_\Omega(\phi)|V|.$$

Proof: Fix $y\,\epsilon\Omega$. Then, using the Cauchy–Schwarz inequality,

$$|T_\phi(V)(y)| \leq \int_\Omega |V(x)||\phi(y-x)| \ d(\text{vol}_x)$$

$$= \int_\Omega |V(x)\{\phi(y-x)\}^{1/2}| \ |\{\phi(y-x)\}^{1/2}| \ d(\text{vol}_x)$$

$$\leq \left(\int_\Omega |V(x)|^2 |\phi(y-x)| \ d(\text{vol}_x) \right)^{1/2} \left(\int_\Omega |\phi(y-x)| \ d(\text{vol}_x) \right)^{1/2}$$

$$\leq (N_\Omega(\phi))^{1/2} \left(\int_\Omega |\phi(y-x)| \ |V(x)|^2 \ d(\text{vol}_x) \right)^{1/2}. \tag{7.1}$$

We square both sides, integrate and use Fubini's theorem to get

$$\int_\Omega |T_\phi(V)(y)|^2 \ d(\text{vol}_y) \leq N_\Omega(\phi) \int_\Omega \int_\Omega |\phi(y-x)||V(x)|^2 \ d(\text{vol}_x) \ d(\text{vol}_y)$$

$$= N_\Omega(\phi) \int_\Omega |V(x)|^2 \left(\int_\Omega |\phi(y-x)| \ d(\text{vol}_y) \right) d(\text{vol}_x)$$

$$\leq N_\Omega(\phi)^2 \int_\Omega |V(x)|^2 \ d(\text{vol}_x). \tag{7.2}$$

Taking square roots, we get

$$|T_\phi(V)| \leq N_\Omega(\phi)|V|, \tag{7.3}$$

and conclude that T_ϕ is a bounded operator whose norm is at most $N_\Omega(\phi)$, as claimed.

B. Proof of (1)

Define the *optical size* of Ω, written $OS(\Omega)$, to be the number

$$OS(\Omega) = \max_y \int_\Omega 1/|y-x|^2 \ d(\text{vol}_x), \tag{7.4}$$

where the maximum is taken over all points $y \in R^3$. The integral just above can be taken as a measure of the effort required to optically scan the domain Ω from the location y; the optical size of Ω is the maximum effort required to scan it from any location.

Then, in the language of Lemma 3,

$$BS(V)(y) = (1/4\pi) \int_\Omega V(x) \times (y-x)/|y-x|^3 \ d(\text{vol}_x)$$

$$= T_{\phi_0}(V)(y), \tag{7.5}$$

where

$$\phi_0(y-x) = (1/4\pi)(1/|y-x|^2). \tag{7.6}$$

The lemma yields immediately that, for $V \in VF(\Omega)$,

$$|BS(V)| \leq (1/4\pi)OS(\Omega)|V|, \tag{7.7}$$

and we conclude that $BS:VF(\Omega) \to VF(\Omega)$ is a bounded operator.

Now let $\overline{\mathrm{VF}}(\Omega)$ denote the L^2 completion of the space $\mathrm{VF}(\Omega)$; we will refer to the elements of $\overline{\mathrm{VF}}(\Omega)$ as L^2 *vector fields*.

Then we can, and do, extend the Biot–Savart operator to a bounded operator,

$$\mathrm{BS}{:}\overline{\mathrm{VF}}(\Omega){\rightarrow}\overline{\mathrm{VF}}(\Omega), \tag{7.8}$$

with the same bound as above.

This completes the proof of part (1).

C. Proof of (2)

To prove that the Biot–Savart operator is compact, we use two standard facts from functional analysis. First is the fact that for any compact domain Ω, if $\phi(x)$ is continuous on R^3, then the integral operator

$$(T_\phi f)(y) = \int_\Omega \phi(y-x) f(x) \ \mathrm{d}(\mathrm{vol}_x) \tag{7.9}$$

defines a compact operator on $L^2(\Omega)$; see Zimmer,[48] Theorem 3.1.5 on p. 53. It is stated there only for operators on scalar-valued functions, but the extension to vector-valued ones, using the definition given in Lemma 9.3, is trivial.

Second is the fact that the norm-limit of compact operators is compact; see Zimmer,[48] Lemma 3.1.3 on p. 52.

Now let

$$\phi_N(x) = \begin{cases} N^2/4\pi & \text{if } |x| \leq 1/N \\ 1/(4\pi|x|^2) & \text{if } |x| \geq 1/N. \end{cases} \tag{7.10}$$

Note that ϕ_N is a continuous function, and that

$$N_\Omega(\phi_0 - \phi_N) = \max_y \int_\Omega |\varphi_0(y-x) - \varphi_N(y-x)| \ \mathrm{d}(\mathrm{vol}_x)$$

$$\leq (1/4\pi) \int_{|x| \leq 1/N} ((1/|x|^2) - N^2) \ \mathrm{d}(\mathrm{vol}_x)$$

$$\leq (1/4\pi) \int_{|x| \leq 1/N} (1/|x|^2) \ \mathrm{d}(\mathrm{vol}_x) = 1/N. \tag{7.11}$$

By the first functional analysis fact, T_{ϕ_N} is a compact operator from $\overline{\mathrm{VF}}(\Omega)$ to $\overline{\mathrm{VF}}(\Omega)$. By our Lemma, we see that as T_{ϕ_N} converges in norm to T_{ϕ_0}, the Biot–Savart operator, as $N{\rightarrow}\infty$. Using the second functional analysis fact, we conclude that $\mathrm{BS}{:}\overline{\mathrm{VF}}(\Omega){\rightarrow}\overline{\mathrm{VF}}(\Omega)$ is a compact operator.

This completes the proof of part (2).

D. Proof of (3)

It is easy to see why the Biot–Savart operator is self-adjoint.

Suppose that V_1 and V_2 are smooth vector fields defined on Ω. Then

$$\langle V_1, \mathrm{BS}(V_2)\rangle = \int_\Omega V_1(y)\cdot \mathrm{BS}(V_2)(y)\ \mathrm{d}(\mathrm{vol}_y)$$

$$= \int_\Omega V_1(y)\cdot\left[(1/4\pi)\int_\Omega V_2(x)\times(y-x)/|y-x|^3\ \mathrm{d}(\mathrm{vol}_x)\right]\mathrm{d}(\mathrm{vol}_y)$$

$$= (1/4\pi)\int_{\Omega\times\Omega} V_1(y)\times V_2(x)\cdot(y-x)/|y-x|^3\ \mathrm{d}(\mathrm{vol}_x)\ \mathrm{d}(\mathrm{vol}_y)$$

$$= (1/4\pi)\int_{\Omega\times\Omega} V_2(x)\times V_1(y)\cdot(x-y)/|x-y|^3\ \mathrm{d}(\mathrm{vol}_y)\ \mathrm{d}(\mathrm{vol}_x)$$

$$= \langle V_2, \mathrm{BS}(V_1)\rangle. \tag{7.12}$$

It is a straightforward exercise to check that these improper integrals are all convergent.

Thus $\mathrm{BS}:\mathrm{VF}(\Omega)\to\mathrm{VF}(\Omega)$ is a self-adjoint operator, and therefore remains self-adjoint when extended to the L^2 completion $\overline{\mathrm{VF}}(\Omega)$ of $\mathrm{VF}(\Omega)$.

Theorem D is proved.

APPENDIX: THE DECAY RATE OF A(V) AND BS(V) AT INFINITY

In item (9) in our list of standard information from Sec. III A, we asserted that in general, A(V) decays at ∞ like $1/r$ and that BS(V) decays at ∞ like $1/r^2$. In the special case that $V\epsilon\mathrm{K}(\Omega)$, we asserted that A($V$) decays at ∞ like $1/r^2$ and that BS(V) decays at ∞ like $1/r^3$.

We give the proofs here.

The defining formula for the vector potential,

$$\mathrm{A}(V)(y) = (1/4\pi)\int_\Omega V(x)/|y-x|\ \mathrm{d}(\mathrm{vol}_x), \tag{A1}$$

expresses an inverse first power law, with integration over a compact region Ω. It follows immediately that A(V) decays at infinity at least as fast as $1/r$.

When we say that A(V) *decays at infinity at least as fast* as $1/r$, we mean that the product $|\mathrm{A}(V)(y)||y|$ has a finite upper bound on R^3, and likewise for corresponding expressions used below.

The Biot–Savart formula,

$$\mathrm{BS}(V)(y) = (1/4\pi)\int_\Omega V(x)\times(y-x)/|y-x|^3\ \mathrm{d}(\mathrm{vol}_x), \tag{A2}$$

expresses an inverse square law, with integration over a compact region Ω. Again it follows immediately that BS(V) decays at infinity at least as fast as $1/r^2$.

The proof of the faster decay rates when $V\epsilon\,\mathrm{K}(\Omega)$ will be divided into two lemmas.

Lemma 4: *The following are equivalent:*

(1) A(V) *decays at infinity at least as fast as* $1/r^2$;
(2) BS(V) *decays at infinity at least as fast as* $1/r^3$;
(3) $\int_\Omega V\mathrm{d}(\mathrm{vol})=0.$

Proof:

It is an easy exercise to check that conditions (1) and (2) each imply (3). For example, when $|y|$ is very large, we have

$$|y| A(V)(y) \approx (1/4\pi) \int_\Omega V(x) \ d(\text{vol}_x).$$ (A3)

If the integral of V is not zero, then $|y|^2 |A(V)(y)|$ certainly blows up at ∞. Thus condition (1) implies condition (3), and likewise, (2) implies (3).

Suppose now that condition (3) holds. Then,

$$|y|^2 A(V)(y) = (1/4\pi) \int_\Omega |y|^2 V(x)/|y-x| \ d(\text{vol}_x)$$

$$= (1/4\pi) \int_\Omega |y|^2 V(x)/|y-x| \ d(\text{vol}_x) - (1/4\pi) \int_\Omega |y| V(x) \ d(\text{vol}_x)$$

$$= (1/4\pi) \int_\Omega \{(|y^2/|y-x|) - |y|\} V(x) \ d(\text{vol}_x),$$ (A4)

where the integral added on the right-hand side is zero thanks to condition (3).

Now,

$$\{(|y|^2/|y-x|) - |y|\} = \{|y|/|y-x|\} \{|y| - |y-x|\}.$$

The first factor on the right-hand side approaches 1 as $y \to \infty$ because Ω is bounded. The second factor on the right-hand side is $\leqslant |x|$, and hence also bounded. Thus

$$\{(|y|^2/|y-x|) - |y|\}$$

is bounded as $y \to \infty$.

Since $\int_\Omega |V(x)| d(\text{vol}_x)$ is certainly bounded, it follows that $|y|^2 |A(V)(y)|$ is bounded, and hence that $A(V)$ decays at ∞ at least as fast as $1/r^2$. Thus condition (3) implies condition (1), as claimed.

Again suppose that condition (3) holds. Then

$$|y|^3 \ \text{BS}(V)(y) = (1/4\pi) \int_\Omega V(x) \times (y-x) |y|^3/|y-x|^3 \ d(\text{vol}_x)$$

$$= (1/4\pi) \int_\Omega V(x) \times (y-x) |y|^3/|y-x|^3 \ d(\text{vol}_x) - (1/4\pi) \int_\Omega V(x) \times y \ d(\text{vol}_x)$$

$$= (1/4\pi) \int_\Omega V(x) \times \{((y-x) |y|^3/|y-x|^3) - y\} \ d(\text{vol}_x),$$ (A5)

where again the integral added on the right-hand side is zero because of condition (3). Continuing,

$$\{((y-x) |y|^3/|y-x|^3) - y\} = \{y(|y|^3 - |y-x|^3)/|y-x|^3\} - \{x|y|^3/|y-x|^3\}.$$

Processing the first term on the right-hand side,

$$\{y(|y|^3 - |y-x|^3)/|y-x|^3\} = \{y/|y-x|\} \ \{|y| - |y-x|\} \ \{(|y|^2 + |y||y-x| + |y-x|^2)/|y-x|^2\}.$$

The first factor on the right-hand side of this last equation is bounded as $y \to \infty$ because Ω is bounded. The second factor on the right-hand side is $\leqslant |x|$, and hence is also bounded. The third factor on the right-hand side approaches the value 3 as $y \to \infty$, and hence is also bounded. It follows that

$$\{y(|y|^3 - |y-x|^3)/|y-x|^3\}$$

is bounded as $y \to \infty$.

Now the term

$$\{x|y|^3/|y-x|^3\}$$

is certainly bounded as $y \to \infty$, and so we conclude that

$$\{((y-x)|y|^3/|y-x|^3)-y\}$$

is also bounded as $y \to \infty$. From this it follows that

$$|y|^3 \, \mathrm{BS}(V)(y)$$

is bounded for all y, and hence that $\mathrm{BS}(V)(y)$ decays at ∞ at least as fast as $1/r^3$. Thus condition (3) implies condition (2).

This completes the proof of Lemma 4.

Lemma 5: $\int_\Omega V(x) d(\mathrm{vol}_x) = 0$ *for all V in* FK$\oplus$HK$\oplus$HG$\oplus$GG, *but this relation determines a codimension-three subspace of* CG.

Proof:

We begin with the proof that $\int_\Omega V(x) d(\mathrm{vol}_x) = 0$ for all $V \epsilon \mathrm{FK} \oplus \mathrm{HK} = K(\Omega)$.

The argument will be coordinate-wise, so that we can deal with scalar-valued integrals instead of vector-valued ones. So let us write the typical point of Ω as $x = (x_1, x_2, x_3)$, and then write $V(x) = (V_1(x), V_2(x), V_3(x))$.

Then

$$\nabla \cdot (x_1 V) = (\nabla x_1) \cdot V + x_1 (\nabla \cdot V) = (\nabla x_1) \cdot V = V_1, \tag{A6}$$

since V is divergence-free.

Hence,

$$\int_\Omega V_1(x) \ d(\mathrm{vol}_x) = \int_\Omega \nabla \cdot (x_1 V) \ d(\mathrm{vol}_x) = \int_{\partial \Omega} x_1 V \cdot n \ d(\mathrm{area}) = 0, \tag{A7}$$

because V is tangent to $\partial \Omega$.

Of course the same argument holds for V_2 and V_3, so we conclude that

$$\int_\Omega V(x) \ d(\mathrm{vol}_x) = 0, \tag{A8}$$

as claimed.

Now we prove that $\int_\Omega V(x) \ d(\mathrm{vol}_x) = 0$ for all $V \epsilon$ HG$\oplus$GG.

Write $V = \nabla \varphi$ with φ constant on each component of $\partial \Omega$.

We claim that

$$\int_\Omega V \ d(\mathrm{vol}) = \int_\Omega \nabla \varphi \ d(\mathrm{vol}) = \int_{\partial \Omega} \varphi n \ d(\mathrm{area}). \tag{A9}$$

We see this as follows:

Let C be any constant vector. Then,

$$\nabla \cdot (\varphi C) = (\nabla \varphi) \cdot C + \varphi (\nabla \cdot C) = (\nabla \varphi) \cdot C. \tag{A10}$$

Hence

$$\left(\int_\Omega \nabla \varphi \ \mathrm{d}(\mathrm{vol}) \right) \cdot C = \int_\Omega (\nabla \varphi) \cdot C \ \mathrm{d}(\mathrm{vol})$$

$$= \int_\Omega \nabla \cdot (\varphi C) \ \mathrm{d}(\mathrm{vol})$$

$$= \int_{\partial\Omega} (\varphi C) \cdot n \ \mathrm{d}(\mathrm{area})$$

$$= \left(\int_{\partial\Omega} \varphi n \ \mathrm{d}(\mathrm{area}) \right) \cdot C. \tag{A11}$$

Since this is true for all constant vectors C, we must have

$$\int_\Omega \nabla \varphi \ \mathrm{d}(\mathrm{vol}) = \int_{\partial\Omega} \varphi n \ \mathrm{d}(\mathrm{area}), \tag{A12}$$

as claimed.

Now suppose that

$$\partial\Omega = \partial\Omega_1 \cup \cdots \cup \partial\Omega_k \tag{A13}$$

is the decomposition of $\partial\Omega$ into its connected components, and let φ_i denote the constant value of the function φ on the boundary component $\partial\Omega_i$. Then

$$\int_\Omega V \ \mathrm{d}(\mathrm{vol}) = \int_\Omega \nabla \varphi \ \mathrm{d}(\mathrm{vol}) = \int_{\partial\Omega} \varphi n \ \mathrm{d}(\mathrm{area}) = \sum_i \varphi_i \int_{\partial\Omega_i} n \ \mathrm{d}(\mathrm{area}) = 0, \tag{A14}$$

because $\int n \ \mathrm{d}(\mathrm{area})$ over any closed surface in 3-space is always zero.

This completes the proof that $\int_\Omega V(x) \ \mathrm{d}(\mathrm{vol}_x) = 0$ for all $V \epsilon \ \mathrm{HG} \oplus \mathrm{GG}$.

The observation that this relation determines a codimension-three subspace of CG follows directly from the fact that the three constant vector fields $\hat{x}$, $\hat{y}$, and $\hat{z}$ are curly gradients, completing the proof of Lemma 5.

Clearly, Lemmas 4 and 5 imply the faster decay rates of A(V) and BS(V) when $V \epsilon \ \mathrm{K}(\Omega)$, completing our argument.

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