Influence of Geometry and Topology on Helicity

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The *helicity* of a smooth vector field defined on a domain in 3-space is the standard measure of the extent to which the field lines wrap and coil around one another; it plays important roles in fluid mechanics, magnetohydrodynamics and plasma physics. In this report we show how the relation between energy and helicity of a vector field is influenced by the geometry and topology of the domain on which it is defined. In particular, we will see that the standard model for the magnetic field in the Crab Nebula (equivalently, the *spheromak field* of plasma physics) is the unique energy-minimizing divergence-free vector field of given nonzero helicity, defined on and tangent to the boundary of a round ball, and that the essential features of this energy-minimizing field persist even as the domain changes topological type. We will also see that when volume-preserving deformation of domain is permitted, the spheromak field is not the absolute energy-minimizing field with given helicity; instead, the round ball on which it is defined can be dimpled in at the poles and expanded out at the equator to further decrease the field energy while preserving helicity. Our numerical computations suggest that this volume-preserving, helicity-preserving, energy-decreasing deformation of domain and field converges to a singular domain, in which the north and south poles have been pressed together at the center, along with a corresponding singular field.

1. Two fundamental problems

We organize this report by focusing on two fundamental problems:

1. Minimize energy among all divergence-free vector fields of given nonzero helicity, defined on and tangent to the boundary of a given domain.
2. Find the above minimum over all domains of given volume.

Such energy-minimizing vector fields provide models for stable force-free magnetic fields in gaseous nebulae and laboratory plasmas, while the search for them seems to bring out some of the deepest and most useful mathematics connected with helicity.

2. Helicity and writhing number

The *helicity* $\mathrm{H}(V)$ of a smooth (meaning C^∞) vector field V on the domain Ω in 3-space, defined by the formula

$$\mathrm{H}(V) = \frac{1}{4\pi} \int_{\Omega\times\Omega} V(x) \times V(y) \cdot \frac{x-y}{|x-y|^3}\, d\,\mathrm{vol}_x\ d\,\mathrm{vol}_y,$$

is the standard measure of the extent to which the field lines wrap and coil around one another. It was introduced in [*Woltjer*, 1958] and named in [*Moffatt*, 1969].

The *writhing number* $\mathrm{Wr}(K)$ of a smooth, arc-length-parametrized curve K in 3-space, defined by the formula

$$\mathrm{Wr}(K) = \frac{1}{4\pi} \int_{K\times K} \frac{dx}{ds} \times \frac{dy}{dt} \cdot \frac{x-y}{|x-y|^3}\, ds\, dt,$$

is the standard measure of the extent to which the curve wraps and coils around itself. It was introduced in [*Călugăreanu*, 1959-61] and named in [*Fuller*, 1971], and has proved its importance for molecular biologists in the study of knotted duplex DNA, and of the enzymes which affect it.

Clearly, writhing number for knots is the analogue of helicity for vector fields. Both formulas above are variants of the integral formula of [*Gauss*, 1833] for the linking number of two disjoint closed space curves.

3. Relation between helicity and writhing number

A useful formula connecting the helicity of vector fields to the writhing of knots appears in [*Berger and Field*, 1984].

Let V be a vector field defined in a tube about a knot K, orthogonal to the cross-sectional disks, with length depending only on distance from K. Such a vector field is always divergence-free.

Then

$$\mathrm{H}(V) = \mathrm{Flux}(V)^2\ \mathrm{Wr}(K).$$

Here, $\mathrm{Flux}(V)$ denotes the flux of V through any of the cross-sectional disks. We also refer the reader to the two papers of [*Moffatt and Ricca*, 1992] for related results.

4. How the geometry of the domain influences helicity

All the numbered theorems in this report are due to the authors, and may be found, together with their proofs in the papers cited.

THEOREM 1. *Let V be a smooth vector field defined on the compact domain Ω with smooth boundary. Then the helicity $\mathrm{H}(V)$ of V is bounded by*

$$|\,\mathrm{H}(V)| \le R(\Omega)\ \mathrm{E}(V),$$

where $R(\Omega)$ is the radius of a ball with the same volume as Ω and $\mathrm{E}(V) = \int_\Omega V \cdot V \, d\,\mathrm{vol}$ is the energy of V.

This upper bound is not sharp, but it is the right order of magnitude. For example, the model for the magnetic field in the Crab Nebula is a vector field V on a round ball Ω with helicity greater than one-fifth of the asserted upper bound. Sharp upper bounds will be obtained by the spectral methods discussed below.

THEOREM 2. *The helicity of a* unit *vector field V defined on the compact domain Ω is bounded by*

$$|\,\mathrm{H}(V)| \leq \frac{1}{2}\,\mathrm{vol}(\Omega)^{4/3}.$$

This theorem, together with the formula of Berger and Field given above, yields an upper bound for the writhing of a DNA strand in terms of its length L and thickness $2R$:

THEOREM 3.

$$|\,\mathrm{Wr}(K)| \leq \frac{1}{4}\left(\frac{L}{R}\right)^{4/3}.$$

Similar bounds have been obtained independently in [*Buck and Simon*, 1998], and also in [*Freedman and He*, 1991].

5. Magnetic fields and helicity

Start with a vector field V on the domain Ω, regard it as a current distribution, and use the Biot-Savart Law to compute its magnetic field:

$$\mathrm{BS}(V)(y) = \frac{1}{4\pi}\int_\Omega V(x) \times \frac{y-x}{|y-x|^3}\, d\,\mathrm{vol}_x \,.$$

The helicity of V can then be expressed as the integrated dot product of V with its magnetic field $\mathrm{BS}(V)$:

$$\begin{aligned}\mathrm{H}(V) &= \frac{1}{4\pi}\int_{\Omega\times\Omega} V(x) \times V(y) \cdot \frac{x-y}{|x-y|^3}\, d\,\mathrm{vol}_x \; d\,\mathrm{vol}_y \\ &= \int_\Omega V(y) \cdot \frac{1}{4\pi}\int_\Omega V(x) \times \frac{y-x}{|y-x|^3}\, d\,\mathrm{vol}_x \; d\,\mathrm{vol}_y \\ &= \int_\Omega V \cdot \mathrm{BS}(V)\, d\,\mathrm{vol}\,.\end{aligned}$$

6. A general point of view

Let Ω be a compact domain in 3-space with smooth boundary. Let $\mathrm{VF}(\Omega)$ denote the set of all smooth vector fields V on Ω. Then $\mathrm{VF}(\Omega)$ is itself an infinite-dimensional vector space.

Define an inner product on $\mathrm{VF}(\Omega)$ by the formula

$$\langle V, W\rangle = \int_{\Omega} V \cdot W \, d\,\mathrm{vol}\,.$$

Although the magnetic field $\mathrm{BS}(V)$ is well-defined throughout all of 3-space, we will restrict it to Ω; thus the Biot-Savart Law provides an operator

$$\mathrm{BS} : \mathrm{VF}(\Omega) \to \mathrm{VF}(\Omega).$$

Using the above inner product notation, our formula for the helicity of V can be written

$$\mathrm{H}(V) = \langle V, \mathrm{BS}(V)\rangle.$$

7. The Modified Biot-Savart Operator

Let $\mathrm{K}(\Omega)$ denote the subspace of $\mathrm{VF}(\Omega)$ consisting of all smooth divergence-free vector fields defined on Ω and tangent to its boundary.

Start with a vector field V in $\mathrm{K}(\Omega)$ and compute its magnetic field, $\mathrm{BS}(V)$. Restrict $\mathrm{BS}(V)$ to Ω and subtract a gradient vector field so as to keep it divergence-free while making it tangent to $\partial\Omega$. Call the resulting vector field $\mathrm{BS}'(V)$. The Hodge Decomposition Theorem in the Appendix tells us that the gradient vector fields on Ω form the orthogonal complement of $\mathrm{K}(\Omega)$; hence $\mathrm{BS}'(V)$ can be viewed as the orthogonal projection of $\mathrm{BS}(V)$ back into $\mathrm{K}(\Omega)$.

The *modified Biot-Savart operator*

$$\mathrm{BS}' : \mathrm{K}(\Omega) \to \mathrm{K}(\Omega),$$

will play a leading role in our story.

The helicity of a vector field V in $K(\Omega)$ is given by

$$\mathrm{H}(V) = \left\langle V, \mathrm{BS}'(V)\right\rangle,$$

since $\mathrm{BS}(V)$ and $\mathrm{BS}'(V)$ differ by a gradient vector field, which as we just noted is orthogonal in the inner product structure of $\mathrm{VF}(\Omega)$ to any vector field V in $\mathrm{K}(\Omega)$.

8. Spectral methods

From now on, we focus on vector fields which are divergence-free and tangent to the boundary of their domain, that is, on the subspace $\mathrm{K}(\Omega)$ of $\mathrm{VF}(\Omega)$, and on the modified Biot-Savart operator $\mathrm{BS}' : \mathrm{K}(\Omega) \to \mathrm{K}(\Omega)$. A standard functional analysis argument yields

THEOREM 4. *The modified Biot-Savart operator* BS' *is a bounded operator, and hence extends to a bounded operator on the L^2 completion of its domain; there it is both compact and self-adjoint.*

The *Spectral Theorem* then promises that BS′ behaves like a real self-adjoint matrix: the L^2 completion of its domain admits an orthonormal basis of eigenfields, in terms of which the operator is "diagonalizable". The eigenfields corresponding to the eigenvalues $\lambda(\Omega)$ of maximum absolute value have maximum helicity for given energy, and we obtain the sharp upper bound

$$|\ \mathrm{H}(V)| \leq |\lambda(\Omega)|\ \ \mathrm{E}(V),$$

for all V in $\mathrm{K}(\Omega)$.

This approach to the study of helicity was initiated in [*Arnold*, 1974] for the setting of closed orientable 3-manifolds. For a corresponding approach via the curl operator on domains in Euclidean space, see [*Yoshida and Giga*, 1990] and [*Laurence and Avellaneda*, 1991].

9. Connection with the curl operator

If the vector field V is divergence-free and tangent to the boundary of its domain Ω, that is, if V is in $\mathrm{K}(\Omega)$, then

$$\nabla \times \mathrm{BS}(V) = V.$$

Since $\mathrm{BS}(V)$ and $\mathrm{BS}'(V)$ differ by a gradient vector field, we also have

$$\nabla \times \mathrm{BS}'(V) = V.$$

If V is an eigenfield of BS′,

$$\mathrm{BS}'(V) = \lambda\, V,$$

then

$$\nabla \times V = \frac{1}{\lambda} V.$$

Thus the eigenvalue problem for BS′ can be converted to an eigenvalue problem for curl on the image of BS′, which means to a system of partial differential equations. Even though we extended BS′ to the L^2 completion of $\mathrm{K}(\Omega)$ in order to apply the spectral theorem, the eigenfields are smooth vector fields in $K(\Omega)$; this follows, thanks to elliptic regularity, because on divergence-free vector fields, the square of the curl is the negative of the Laplacian. Hence these vector fields can be (and are) discovered by solving the above system of PDEs.

10. Explicit computation of energy-minimizing vector fields

We solve $\nabla \times V = (1/\lambda)V$ on the flat solid torus $D^2(a) \times S^1$, where $D^2(a)$ is a disk of radius a and S^1 is a circle of any length; see [*Cantarella et al*, 1997a]. Although this is not a subdomain of 3-space, the solution here is so clear-cut and instructive as to be irresistible.

The eigenvalues of BS′ of largest absolute value are

$$\lambda(D^2(a) \times S^1) = \pm \frac{a}{2.405...},$$

where the denominator is the first positive zero of the Bessel function J_0, and the corresponding eigenfields, discovered in [*Lundquist*, 1951], are

$$V_\lambda = J_1(r/\lambda)\,\hat{\varphi} + J_0(r/\lambda)\,\hat{z},$$

expressed in terms of cylindrical coordinates (r, φ, z) and the Bessel functions J_0 and J_1.

It follows that for any V in $K(D^2(a) \times S^1)$,

$$|\ \mathrm{H}(V)| \leq \frac{a}{2.405..}\ \mathrm{E}(V),$$

with equality for the eigenfield V_λ.

We solve $\nabla \times V = (1/\lambda)V$ on the round ball $B^3(a)$ of radius a in terms of spherical Bessel functions in [*Cantarella et al*, 1998b].

The eigenvalues of BS′ of largest absolute value are

$$\lambda(B^3(a)) = \pm \frac{a}{4.4934...},$$

where the denominator is the first positive zero of the function $(\sin x)/x - \cos x$. The corresponding eigenfields V_λ are Woltjer's models for the magnetic field in the Crab Nebula. In spherical coordinates (r, θ, φ) on a ball of radius $a = 1$,

$$V_\lambda(r, \theta, \varphi) = u(r,\theta)\,\hat{r} + v(r,\theta)\,\hat{\theta} + w(r,\theta)\,\hat{\varphi},$$

where

$$u(r,\theta) = \frac{2\lambda}{r^2}\left(\frac{\sin(r/\lambda)}{r/\lambda} - \cos(r/\lambda)\right)\cos\theta,$$

$$v(r,\theta) = -\frac{1}{r}\left(\frac{\cos(r/\lambda)}{r/\lambda} - \frac{\sin(r/\lambda)}{(r/\lambda)^2} + \sin(r/\lambda)\right)\sin\theta,$$

$$w(r,\theta) = \frac{1}{r}\left(\frac{\sin(r/\lambda)}{r/\lambda} - \cos(r/\lambda)\right)\sin\theta.$$

The values $\lambda = \pm 1/4.4934...$ make both $u(r, \theta)$ and $w(r, \theta)$ vanish when r=1, that is, at the boundary of the ball. As a consequence, the vector field V_λ is tangent to the boundary of the ball, and directed there along the meridians of longitude.

It follows that for any V in $K(B^3(a))$,

$$|\ \mathrm{H}(V)| \leq \frac{a}{4.4934...}\ \mathrm{E}(V),$$

with equality for the eigenfield V_λ.

Compare this with the rough upper bound from Theorem 1:

$$|\, \mathrm{H}(V)| \leq a\ \mathrm{E}(V).$$

11. The isoperimetric problem

We focus now on our second fundamental problem, a special case of which was considered in [*Chui and Moffatt*, 1995]:

Minimize energy among all divergence-free vector fields of given nonzero helicity, defined on and tangent to the boundary of all domains of given volume in 3-space.

When the domain Ω is fixed, the largest eigenvalue $\lambda(\Omega)$ of the modified Biot-Savart operator BS' is the largest possible value of the Rayleigh quotient:

$$\lambda(\Omega) = \max_V \frac{\mathrm{H}(V)}{\mathrm{E}(V)} = \max_V \frac{\langle V, \mathrm{BS}'(V)\rangle}{\langle V, V\rangle}.$$

To maximize $\lambda(\Omega)$ among all domains of given volume, we want to take the "first derivative" of this quotient as the domain varies and set it equal to zero. This leads us to seek first variation formulas for helicity and energy.

12. First Variation Formulas

Suppose the domain Ω is subject to a smooth volume-preserving deformation $h_t : \Omega \to \Omega_t$, with h_0 the identity, and initial velocity the divergence-free vector field W defined by $W(x) = \frac{d}{dt}|_{t=0}\, h_t(x)$.

Choose a vector field V in $\mathrm{K}(\Omega)$, and let $V_t = (h_t)_* V$ be its push-forward to a vector field on the domain Ω_t. In other words, let V_t be frozen into the domain Ω_t as it deforms.

THEOREM 5. *The helicity* $\mathrm{H}(V_t)$ *is independent of* t.

THEOREM 6. *The first variation of the energy of* V_t*, calculated at* $t = 0$*, is given by*

$$\delta E(V) = 2\langle V \times (\nabla \times V), W\rangle - \int_{\partial\Omega} |V|^2\, (W \cdot n)\, d\,\mathrm{area}\,.$$

THEOREM 7. *The first variation of the largest eigenvalue* $\lambda(\Omega_t)$ *of* BS' *on* Ω_t*, calculated at* $t = 0$*, satisfies the inequality*

$$\delta\lambda(\Omega) \geq \lambda(\Omega)\, \frac{\int_{\partial\Omega} |V_\lambda|^2\, (W \cdot n)\, d\,\mathrm{area}}{\int_\Omega |V_\lambda|^2\, d\,\mathrm{vol}}.$$

where V_λ *is a corresponding eigenfield.*

The inequality appears only in the case that the largest eigenvalue has multiplicity > 1. This can certainly happen: when Ω is a round ball the largest eigenvalue has multiplicity 3. When this eigenvalue is simple, the inequality can be replaced by an equality.

13. Constraints on any optimal domain

The first variation formula in Theorem 7 leads in turn to

THEOREM 8. *Suppose the vector field V defined on the compact, smoothly bounded domain Ω minimizes energy among all divergence-free vector fields of given nonzero helicity, defined on and tangent to the boundary of all such domains of given volume in 3-space. Then*

1. *$|V|$ is a nonzero constant on $\partial\Omega$.*
2. *All the components of $\partial\Omega$ are tori.*
3. *The orbits of V are geodesics on $\partial\Omega$.*

Thus, *no* smooth simply connected domain is optimal in the above sense. In principle, one could have a smooth optimal domain in the shape, say, of a solid torus. But we believe that *there are no smooth optimal domains at all,* regardless of topological type, and that the true optimizer looks like the singular domain shown in the next section.

14. The search for optimal domains

Suppose we begin with the spheromak field V_λ which minimizes energy for given nonzero helicity on a round ball Ω, as discussed and pictured in section 10.

We seek a volume-preserving deformation of Ω which increases $\lambda(\Omega)$, guided by our inequality

$$\delta\lambda(\Omega) \geq \lambda(\Omega)\,\frac{\int_{\partial\Omega} |V_\lambda|^2 (W\cdot n)\, d\,\mathrm{area}}{\int_\Omega |V_\lambda|^2\, d\,\mathrm{vol}}.$$

We maximize the right hand side by choosing

$$W\cdot n = |V_\lambda|^2 - \text{average value of } |V_\lambda|^2 \text{ on } \partial\Omega.$$

Then we imagine a volume-preserving deformation of Ω whose initial velocity field W has this preassigned normal component along the boundary. The deformation begins by dimpling Ω inwards near the poles and bulging it outwards near the equator, making the ball look somewhat like an apple.

At each stage Ω_t of the deformation, consider a vector field V_t which minimizes energy for given helicity on Ω_t, with $V_0 = V_\lambda$, and which determines the normal component $W_t \cdot n$ of the deformation velocity field W_t along the boundary in the same way as at the beginning:

$$W_t\cdot n = |V_t|^2 - \text{average value of } |V_t|^2 \text{ on } \partial\Omega_t.$$

Such a deformation tries to follow a path of steepest ascent for the largest eigenvalue $\lambda(\Omega_t)$ of the modified Biot-Savart operator.

We believe that this procedure will continue to dimple the apple inwards at the poles and bulge it outwards at the equator, until it reaches roughly the shape pictured below, which then maximizes the largest eigenvalue $\lambda(\Omega)$ of the modified Biot-Savart operator among all domains of given volume. We can think of this singular domain either as an extreme apple, in which the north and south poles have been pressed together, or as an extreme solid torus, in which the hole has been shrunk to a point. We also show in the sketch the expected appearance of the energy-minimizing vector field. The domain curiously resembles the NSTX (National Spherical Torus Experiment) containment device currently under construction at the Princeton Plasma Physics Laboratory.

Comparison of this picture with those of the energy-minimizers on the flat solid torus and on the round ball, given earlier, shows that we expect the common underlying pattern to persist even as the domain becomes singular, with the field in each case tangent to a family of nested tori with a single core curve.

A computational search for this singular optimal domain and the energy-minimizing vector field on it is at present under way, guided by a discrete version of the evolution described above.

APPENDIX. THE HODGE DECOMPOSITION THEOREM

How Domain Topology Influences Vector Calculus

In a multivariable calculus course, we are taught that the topology of the underlying domain affects the calculus of vector fields defined on it. For example, we learn that to test whether a vector field is the gradient of a function, we must take its curl and see if it is zero. If the curl is not zero, then the vector field is certainly not a gradient. If the curl is zero and the domain is simply connected, we learn that the vector field is a gradient. But if the curl is zero and the domain is not simply connected, then we learn that the vector field may or may not be a gradient, and that further tests are required.

The Hodge Decomposition Theorem for vector fields on domains in 3-space provides a more sophisticated level of control over this same subject.

The following two questions help to set the mood.

Question 1. *Is there a nonzero vector field V on the domain which is divergence-free, curl-free and tangent to the boundary?*

Question 2. *Is there a nonzero gradient vector field V on the domain which is divergence-free and orthogonal* *to the boundary?*

Domain	Answers to Question	
	1	2
Ball	No	No
Solid torus	Yes	No
Spherical shell	No	Yes
Toroidal shell	Yes	Yes

The Hodge Decomposition Theorem

Let Ω be a compact domain with smooth boundary in 3-space.

The following is arguably the single most useful expression of the interplay between the topology of the domain Ω, the traditional calculus of vector fields defined on this domain, and the inner product structure on $\mathrm{VF}(\Omega)$ defined in section 6 by the formula $\langle V, W\rangle = \int_\Omega V \cdot W \, d\,\mathrm{vol}$.

[*Blank-Friedrichs-Grad*, 1957] and [*Schwarz*, 1995] are good references; a detailed treatment and proof of this theorem in the form given below appears in our paper [*Cantarella et al*, 1997b].

HODGE DECOMPOSITION THEOREM. *We have a direct sum decomposition of* $\mathrm{VF}(\Omega)$ *into five mutually orthogonal subspaces,*

$$\mathrm{VF}(\Omega) = \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG},$$

with

$$\begin{aligned} \ker \mathrm{curl} &= \phantom{\mathrm{FK} \oplus{}} \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG} \\ \mathrm{image}\,\mathrm{grad} &= \phantom{\mathrm{FK} \oplus \mathrm{HK} \oplus{}} \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG} \\ \mathrm{image}\,\mathrm{curl} &= \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG} \\ \ker \mathrm{div} &= \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG} \end{aligned}$$

where

$$\begin{aligned} \mathrm{FK} &= \{\nabla \cdot V = 0,\ V \cdot n = 0,\ \text{all interior fluxes } = 0\}, \\ \mathrm{HK} &= \{\nabla \cdot V = 0, \nabla \times V = 0, V \cdot n = 0\}, \\ \mathrm{CG} &= \{V = \nabla\varphi, \nabla \cdot V = 0, \text{all boundary fluxes} = 0\}, \\ \mathrm{HG} &= \{V = \nabla\varphi, \nabla \cdot V = 0, \varphi \text{ loc. constant on } \partial\Omega\}, \\ \mathrm{GG} &= \{V = \nabla\varphi,\ \varphi|_{\partial\Omega} = 0\}, \end{aligned}$$

and furthermore,

$$\begin{aligned} \mathrm{HK} &\cong H_1(\Omega; \mathbf{R}) \cong H_2(\Omega, \partial\Omega; \mathbf{R}) \\ &\cong \mathbf{R}^{\text{genus of } \partial\Omega}. \\ \mathrm{HG} &\cong H_2(\Omega; \mathbf{R}) \cong H_1(\Omega, \partial\Omega; \mathbf{R}) \\ &\cong \mathbf{R}^{(\#\text{ components of } \partial\Omega) - (\#\text{ components of } \Omega)}. \end{aligned}$$

We need to explain the meanings of the conditions which appear in the statement of this theorem.

The outward pointing unit vector field orthogonal to $\partial\Omega$ is denoted by n, so the condition $V \cdot n = 0$ indicates that V is tangent to the boundary of Ω.

Let Σ stand generically for any smooth surface in Ω with $\partial\Sigma \subset \partial\Omega$. Orient Σ by picking one of its two unit normal vector fields n. Then, for any vector field V on Ω, the *flux* of V through Σ is the value of the integral $\Phi = \int_\Sigma V \cdot n \, d\,\text{area}$.

If V is divergence-free and tangent to $\partial\Omega$, then the value of this flux depends only on the homology class of Σ in the relative homology group $H_2(\Omega, \partial\Omega; \mathbf{R})$. For example, if Ω is an n-holed solid torus, then there are disjoint oriented cross-sectional disks $\Sigma_1, \ldots, \Sigma_n$, positioned so that cutting Ω along these disks will produce a simply-connected region. The fluxes $\Phi_1, \ldots, \Phi_n$ of V through these disks determine the flux of V through any other cross-sectional surface.

If the flux of V through every smooth surface Σ in Ω with $\partial\Sigma \subset \partial\Omega$ vanishes, we say *all interior fluxes* $= 0$. Thus the subspace of vector fields V in $\mathrm{VF}(\Omega)$ which have

$$\nabla \cdot V = 0,\ V \cdot n = 0, \text{ and all interior fluxes} = 0,$$

is called the subspace FK of *fluxless knots.*

The subspace HK of vector fields V in $\mathrm{VF}(\Omega)$ with

$$\nabla \cdot V = 0,\ \nabla \times V = 0,\ V \cdot n = 0,$$

called *harmonic knots*, is isomorphic to the absolute homology group $H_1(\Omega; \mathbf{R})$ and also by Poincaré duality to the relative homology group $H_2(\Omega, \partial\Omega; \mathbf{R})$. It is thus a finite-dimensional vector space, with dimension equal to the (total) genus of $\partial\Omega$.

The orthogonal direct sum of these two subspaces,

$$\mathrm{K}(\Omega) = \mathrm{FK} \oplus \mathrm{HK},$$

is the subspace of $\mathrm{VF}(\Omega)$ mentioned earlier, consisting of all divergence-free vector fields defined on Ω and tangent to its boundary.

If V is a vector field defined on Ω, we will say that *all boundary fluxes of V are zero* if the flux of V through each component of $\partial\Omega$ is zero. The subspace of V in $\mathrm{VF}(\Omega)$ with

$$V = \nabla\varphi,\ \nabla \cdot V = 0, \text{ all boundary fluxes} = 0$$

is called the subspace CG of *curly gradients*, because these are the only gradients which lie in the image of curl.

The subspace HG of *harmonic gradients* consists of all V in $\mathrm{VF}(\Omega)$ such that

$$V = \nabla\varphi,\ \nabla \cdot V = 0,\ \varphi \text{ locally constant on } \partial\Omega,$$

meaning that φ is constant on each component of $\partial\Omega$. This subspace is isomorphic to the absolute homology group $H_2(\Omega; \mathbf{R})$ and also, via Poincaré duality, to the relative homology group $H_1(\Omega, \partial\Omega; \mathbf{R})$, and is hence a finite-dimensional vector space, with dimension equal to the number of components of $\partial\Omega$ minus the number of components of Ω.

The definition of the subspace GG of *grounded gradients*, which consists of all V in VF(Ω) such that

$$V = \nabla\varphi,\ \varphi|_{\partial\Omega} = 0,$$

is self-explanatory.

We refer the reader to [*Cantarella et al*, 1997b] for a thorough treatment of the Hodge Decomposition Theorem and a variety of applications to boundary value problems for vector fields.

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