Isoperimetric problems for the helicity of vector fields and the Biot–Savart and curl operators

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The *helicity* of a smooth vector field defined on a domain in three-space is the standard measure of the extent to which the field lines wrap and coil around one another. It plays important roles in fluid mechanics, magnetohydrodynamics, and plasma physics. The isoperimetric problem in this setting is to maximize helicity among all divergence-free vector fields of given energy, defined on and tangent to the boundary of all domains of given volume in three-space. The *Biot–Savart operator* starts with a divergence-free vector field defined on and tangent to the boundary of a domain in three-space, regards it as a distribution of electric current, and computes its magnetic field. Restricting the magnetic field to the given domain, we modify it by subtracting a gradient vector field so as to keep it divergence-free while making it tangent to the boundary of the domain. The resulting operator, when extended to the L^2 completion of this family of vector fields, is compact and self-adjoint, and thus has a largest eigenvalue, whose corresponding eigenfields are smooth by elliptic regularity. The isoperimetric problem for this modified Biot–Savart operator is to maximize its largest eigenvalue among all domains of given volume in three-space. The *curl operator*, when restricted to the image of the modified Biot–Savart operator, is its inverse, and the isoperimetric problem for this restriction of the curl is to minimize its smallest positive eigenvalue among all domains of given volume in three-space. These three isoperimetric problems are equivalent to one another. In this paper, we will derive the first variation formulas appropriate to these problems, and use them to constrain the nature of any possible solution. For example, suppose that the vector field V, defined on the compact, smoothly bounded domain Ω, maximizes helicity among all divergence-free vector fields of given nonzero energy, defined on and tangent to the boundary of all such domains of given volume. We will show that (1) $|V|$ is a nonzero constant on the boundary of each component of Ω; (2) all the components of $\partial\Omega$ are tori; and (3) the orbits of V are geodesics on $\partial\Omega$. Thus, among smooth simply connected domains, *none* are optimal in the above sense. In principal, one could have a smooth optimal domain in the shape, say, of a solid torus. However, we believe that there are *no smooth optimal domains at all*, regardless of topological type, and that the true optimizer looks like the singular domain presented in this paper, which we can think of either as an extreme apple, in which the north and south poles have been pressed together, or as an extreme solid torus, in which the hole has been shrunk to a point. A computational search for this singular optimal domain and the helicity-maximizing vector field on it is at present under way, guided by the first variation formulas in this paper. © *2000 American Institute of Physics.*
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I. INTRODUCTION

Let Ω be a compact domain in three-space with smooth boundary $\partial\Omega$; ''smooth'' for us always means of class C^∞. We allow both Ω and $\partial\Omega$ to be disconnected.

Let VF(Ω) be the set of all smooth vector fields V on Ω. Then VF(Ω) is an infinite-dimensional vector space, on which we use the L^2 inner product $\langle V,W\rangle = \int_\Omega V\cdot W\, d(\mathrm{vol})$.

The *helicity* $H(V)$ of the vector field V on Ω, defined by the formula

$$H(V)=(1/4\pi)\int_{\Omega\times\Omega} V(x)\times V(y)\cdot(x-y)/|x-y|^3 d(\mathrm{vol}_x)d(\mathrm{vol}_y),$$

was introduced by Woltjer[1] in 1958 and named by Moffatt[2] in 1969. The formula itself is a variation on Gauss' integral formula[3] for the linking number of two closed space curves, which dates back to 1833.

To help understand the formula for helicity, think of V as a distribution of electric current, and use the Biot–Savart law to compute its magnetic field, BS(V):

$$\mathrm{BS}(V)(y)=(1/4\pi)\int_{\Omega} V(x)\times(y-x)/|y-x|^3 d(\mathrm{vol}_x).$$

Although the magnetic field BS(V) is well defined throughout all of three-space, we will restrict it to the domain Ω and thus view the Biot–Savart law as providing an operator

$$\mathrm{BS}:\mathrm{VF}(\Omega)\to\mathrm{VF}(\Omega).$$

The relation between helicity and the Biot–Savart operator is as follows:

$$\begin{aligned}
H(V)&=(1/4\pi)\int_{\Omega\times\Omega} V(x)\times V(y)\cdot(x-y)/|x-y|^3 d(\mathrm{vol}_x)d(\mathrm{vol}_y)\\
&=\int_\Omega V(y)\cdot\left[(1/4\pi)\int_\Omega V(x)\times(y-x)/|y-x|^3 d(\mathrm{vol}_x)\right]d(\mathrm{vol}_y)\\
&=\int_\Omega V(y)\cdot\mathrm{BS}(V)(y)\,d(\mathrm{vol}_y)\\
&=\int_\Omega V\cdot\mathrm{BS}(V)\,d(\mathrm{vol}),
\end{aligned}$$

so the helicity of V is just the L^2 inner product of V and BS(V),

$$H(V)=\langle V,\ \mathrm{BS}(V)\rangle.$$

In this paper, we will mainly be interested in divergence-free vector fields which are defined on and tangent to the boundary of the domain Ω. They form a subspace K(Ω) of VF(Ω),

$$\mathrm{K}(\Omega)=\{V\in\mathrm{VF}(\Omega):\ \nabla\cdot V=0,\ \ V\cdot n=0\},$$

where n is the unit outward normal vector field to $\partial\Omega$. These vector fields are often regarded as the fluid analogs of knots and links.

Recall the modification of the Biot–Savart operator described in the abstract on the first page. We start with a divergence-free vector field V, defined on and tangent to the boundary of Ω, thus an element of K(Ω). We compute its magnetic field BS(V) and restrict it to Ω. Then we subtract an appropriate gradient vector field from BS(V) so that the resulting vector field lies in K(Ω); see

Sec. II A for the Hodge Decomposition theorem. To say it another way, we take the L^2 orthogonal projection of BS(V) back into K(Ω). In this way we define the *modified Biot–Savart operator*

$$\mathrm{BS}'\colon \mathrm{K}(\Omega)\to\mathrm{K}(\Omega).$$

Just as the Biot–Savart operator BS is related to helicity by the formula

$$H(V)=\langle V, \mathrm{BS}(V)\rangle$$

for any $V\in\mathrm{VF}(\Omega)$, so the modified Biot–Savart operator BS′ is related to helicity by the formula

$$H(V)=\langle V, \mathrm{BS}'(V)\rangle$$

for any $V\in\mathrm{K}(\Omega)$. The second formula follows from the first, since $\mathrm{BS}'(V)$ differs from BS(V) by a gradient vector field, which is L^2 orthogonal to V if $V\in\mathrm{K}(\Omega)$.

Since we are focusing on divergence-free vector fields which are tangent to the boundary of their domain of definition, it is this second formula for helicity which plays a central role in the present paper.

The modified Biot–Savart operator BS′, when extended to the L^2 completion of its domain K(Ω), is a compact, self-adjoint operator. Applying the spectral theorem and elliptic regularity, we will see that the vector fields V in K(Ω) with maximum helicity for given energy are precisely the eigenfields of BS′ corresponding to its largest eigenvalue $\lambda(\Omega)$, and that for these vector fields we have

$$H(V)=\lambda(\Omega)\ E(V),$$

where $E(V)=\langle V,V\rangle$ is the energy of V. Then for all V in K(Ω) we have

$$H(V)\leqslant\lambda(\Omega)\ E(V).$$

This approach to helicity was pioneered by Arnold[4] in his 1974 study of the asymptotic Hopf invariant for vector fields on closed orientable three-manifolds.

Searching for the largest eigenvalue of BS′ on VF(Ω) might seem to favor vector fields of positive helicity. However, if we reflect the domain Ω through the origin in three-space to obtain the domain Ω^-, and carry along the vector field V on Ω to a vector field V^- on Ω^-, then helicities change sign because the reflection is orientation reversing. That is, $H(V^-)=-H(V)$. Thus the vector fields of negative helicity on Ω reflect through the origin to vector fields of positive helicity on Ω^-, where they get their deserved attention. In particular, for any vector field V on Ω, we have

$$|H(V)|\leqslant\max\{\lambda(\Omega),\lambda(\Omega^-)\}\ E(V).$$

Suppose the domain Ω is subject to a smooth volume-preserving deformation $h_t\colon\Omega\to\Omega_t$, with h_0 the identity, whose initial velocity is the vector field W defined by $W(x)=d/dt|_{t=0}h_t(x)$. By ''volume-preserving,'' we always mean that the volume form is preserved at each point; thus $\nabla\cdot W=0$. We would like to have a first variation formula for the largest eigenvalue $\lambda(\Omega)$ of the modified Biot–Savart operator $\mathrm{BS}'\colon\mathrm{K}(\Omega)\to\mathrm{K}(\Omega)$.

However, as we know from elementary linear algebra, the largest eigenvalue of a smooth one-parameter family of self-adjoint matrices does not always vary smoothly.

We finesse this annoyance as follows. Given a divergence-free vector field V defined on and tangent to the boundary of Ω, consider the *Rayleigh quotient*

$$\lambda(V)=\langle \mathrm{BS}'(V),V\rangle/\langle V,V\rangle=H(V)/E(V).$$

If V happens to be an eigenfield of the modified Biot–Savart operator BS', then $\lambda(V)$ will be the corresponding eigenvalue. The largest eigenvalue $\lambda(\Omega)$ of BS' is the maximum of all the Rayleigh quotients $\lambda(V)$.

Now, given the smooth volume-preserving deformation h_t of Ω defined above, let $V_t=(h_t)_*V$ be the push-forward of V to a vector field on the domain Ω_t. One says that V_t is *frozen* into the domain Ω_t as it deforms. The quantity $\lambda(V_t)$ does vary smoothly, so we define the first variation $\delta\lambda(V)$ of $\lambda(V)$ to be

$$\delta\lambda(V)=d/dt|_{t=0}\lambda(V_t)$$

and seek a formula for $\delta\lambda(V)$.

Since $\lambda(V)=H(V)/E(V)$, it is natural to seek first variation formulas for the helicity $H(V)$ and the energy $E(V)$.

In the following theorems, keep in mind that the vector field V is divergence-free and tangent to the boundary of the domain Ω, and remains frozen in as Ω is subject to a volume-preserving deformation with initial velocity field W.

Theorem A: *The helicity $H(V_t)$ is independent of t.*

This theorem is inspired by Arnold,[4] who showed that for certain divergence-free vector fields V on a compact orientable three-manifold without boundary, the helicity $H(V)$ remains constant when V is carried along by any volume-preserving, orientation-preserving diffeomorphism. We discuss this at the beginning of Sec. III.

Theorem B: *The first variation of energy is given by the formula*

$$\delta E(V)=2\langle V\times(\nabla\times V),\ W\rangle-\int_{\partial\Omega}|V|^2(W\cdot n)\,d(\text{area}).$$

If the domain Ω is again replaced by a compact orientable three-dimensional manifold without boundary, then the second term on the right disappears, and Theorem B reduces to another result of Arnold.[4]

Theorem C: *The first variation of the Rayleigh quotient $\lambda(V)=H(V)/E(V)$ is given by the formula*

$$\delta\lambda(V)=\lambda(V)\,\frac{-2\langle V\times(\nabla\times V),\ W\rangle+\int_{\partial\Omega}|V|^2(W\cdot n)\,d(\text{area})}{\int_\Omega |V|^2\,d(\text{vol})}.$$

If V is an eigenfield of the modified Biot–Savart operator BS', *then*

$$\delta\lambda(V)=\lambda(V)\,\frac{\int_{\partial\Omega}|V|^2(W\cdot n)\,d(\text{area})}{\int_\Omega |V|^2\,d(\text{vol})}.$$

If this eigenfield V corresponds to the largest eigenvalue $\lambda(\Omega)$ *of* BS' *on* Ω, *then*

$$\delta\lambda(\Omega)\geqslant\lambda(\Omega)\,\frac{\int_{\partial\Omega}|V|^2(W\cdot n)\,d(\text{area})}{\int_\Omega |V|^2\,d(\text{vol})}.$$

The inequality appears only in the case that the largest eigenvalue has multiplicity >1. This can certainly happen: when Ω is a round ball the largest eigenvalue has multiplicity 3. When this eigenvalue is simple, the inequality can be replaced by an equality.

The third part of Theorem C plays a key role in proving the next theorem.

Theorem D: *Suppose the vector field V, defined on the compact, smoothly bounded domain Ω, maximizes helicity among all divergence-free vector fields of given nonzero energy, defined on and tangent to the boundary of all such domains of given volume in three-space.*

Then

(1) $|V|$ *is a nonzero constant on the boundary of each component of* Ω.
(2) *All the components of* $\partial\Omega$ *are tori*.
(3) *The orbits of* V *are geodesics on* $\partial\Omega$.

We already mentioned some of the consequences of this result in the abstract.

After proving Theorem A, we will modify its proof to derive a general first variation formula for helicity,

$$\delta H(V) = 2\int_\Omega (\mathrm{BS}(V)\cdot V)(\nabla\cdot W)\, d(\mathrm{vol}),$$

in which the vector field V is, as usual, divergence-free and tangent to the boundary of its domain Ω, but in which the deformation h_t is not required to be volume preserving, and hence in which its initial velocity field W is arbitrary. But we will not use this formula in the paper.

After proving Theorem C, we will describe an alternative first variation formula for the eigenvalues of the modified Biot–Savart operator BS' which appears as an equality rather than an inequality.

For further information about helicity, its mathematical foundations, and the role it plays in fluid mechanics and plasma physics, we refer the reader to the papers of Berger and Field,[5] Moffatt and Ricca,[6,7] and to our papers.[8–14]

II. BACKGROUND

A. The Hodge decomposition theorem

Let Ω be a compact domain with smooth boundary in three-space.

The following theorem is arguably the single most useful expression of the interplay between the topology of the domain Ω, the traditional calculus of vector fields defined on this domain, and the L^2 inner product structure on $\mathrm{VF}(\Omega)$. We will use this result a number of times in the sections to come.

The reader can find a detailed treatment and proof of this theorem in Ref. 9, along with a number of applications to boundary value problems for vector fields.

Hodge decomposition theorem: *We have a direct sum decomposition of* $\mathrm{VF}(\Omega)$ *into five mutually orthogonal subspaces*,

$$\mathrm{VF}(\Omega) = \mathrm{FK}\oplus\mathrm{HK}\oplus\mathrm{CG}\oplus\mathrm{HG}\oplus\mathrm{GG},$$

with

$$\begin{aligned}
\ker\ \mathrm{curl} &= \mathrm{HK}\oplus\mathrm{CG}\oplus\mathrm{HG}\oplus\mathrm{GG},\\
\mathrm{image}\ \mathrm{grad} &= \mathrm{CG}\oplus\mathrm{HG}\oplus\mathrm{GG},\\
\mathrm{image}\ \mathrm{curl} &= \mathrm{FK}\oplus\mathrm{HK}\oplus\mathrm{CG},\\
\ker\ \mathrm{div} &= \mathrm{FK}\oplus\mathrm{HK}\oplus\mathrm{CG}\oplus\mathrm{HG},
\end{aligned}$$

where

$\mathrm{FK} =$ *Fluxless knots* $= \{\nabla\cdot V = 0, V\cdot n = 0, \text{all interior fluxes} = 0\}$,
$\mathrm{HK} =$ *Harmonic knots* $= \{\nabla\cdot V = 0, \nabla\times V = 0, V\cdot n = 0\}$,
$\mathrm{CG} =$ *Curly gradients* $= \{V = \nabla\varphi, \nabla\cdot V = 0, \text{all boundary fluxes} = 0\}$,
$\mathrm{HG} =$ *Harmonic gradients* $= \{V = \nabla\varphi, \nabla\cdot V = 0, \varphi \text{ locally constant on } \partial\Omega\}$,
$\mathrm{GG} =$ *Grounded gradients* $= \{V = \nabla\varphi, \varphi|\partial\Omega = 0\}$,

and furthermore,

$$\mathrm{HK} \cong H_1(\Omega;R) \cong H_2(\Omega,\partial\Omega;R) \cong R^{\text{genus of } \partial\Omega},$$
$$\mathrm{HG} \cong H_2(\Omega;R) \cong H_1(\Omega,\partial\Omega;R) \cong R^{(\#\text{ components of } \partial\Omega)-(\#\text{ components of } \Omega)}.$$

We need to explain the meanings of the conditions which appear in the statement of this theorem.

The outward pointing unit vector field orthogonal to $\partial\Omega$ is denoted by n, so the condition $V\cdot n=0$ indicates that the vector field V is tangent to the boundary of Ω.

Let Σ stand generically for any smooth surface in Ω with $\partial\Sigma \subset \partial\Omega$. Orient Σ by picking one of its two unit normal vector fields n. Then, for any vector field V on Ω, we can define the *flux* of V through Σ to be the value of the integral $\Phi=\int_\Sigma V\cdot n\, d(\text{area})$.

Assume that V is divergence-free and tangent to $\partial\Omega$. Then the value of this flux depends only on the homology class of Σ in the relative homology group $H_2(\Omega,\partial\Omega;Z)$. For example, if Ω is an n-holed solid torus, then there are disjoint oriented cross-sectional disks $\Sigma_1,\ldots,\Sigma_n$, positioned so that cutting Ω along these disks will produce a simply-connected region. The fluxes $\Phi_1,\ldots,\Phi_n$ of V through these disks determine the flux of V through any other cross-sectional surface.

If the flux of V through every smooth surface Σ in Ω with $\partial\Sigma\subset\partial\Omega$ vanishes, we'll say ''*all interior fluxes*$=0$.'' Then

$$\mathrm{FK}=\{V\in \mathrm{VF}(\Omega):\ \nabla\cdot V=0,\ \ V\cdot n=0,\ \ \text{all interior fluxes}=0\}$$

will be the subspace of *fluxless knots*.

The subspace

$$\mathrm{HK}=\{V\in \mathrm{VF}(\Omega):\ \nabla\cdot V=0,\ \nabla\times V=0,\ V\cdot n=0\}$$

of *harmonic knots* is isomorphic to the absolute homology group $H_1(\Omega;R)$ and also, via Poincaré duality, to the relative homology group $H_2(\Omega,\partial\Omega;R)$, and is thus a finite-dimensional vector space, with dimension equal to the genus of $\partial\Omega$.

The orthogonal direct sum of these two subspaces,

$$\mathrm{K}(\Omega)=\mathrm{FK}\oplus\mathrm{HK},$$

is the subspace of VF(Ω) mentioned earlier, consisting of all divergence-free vector fields defined on Ω and tangent to its boundary.

If V is a vector field defined on Ω, we will say that *all boundary fluxes of V are zero* if the flux of V through each component of $\partial\Omega$ is zero. Then

$$\mathrm{CG}=\{V\in \mathrm{VF}(\Omega):\ V=\nabla\varphi,\ \ \nabla\cdot V=0,\ \ \text{all boundary fluxes}=0\}$$

will be called the subspace of *curly gradients* because these are the only gradients which lie in the image of curl.

We define the subspace of *harmonic gradients*,

$$\mathrm{HG}=\{V\in \mathrm{VF}(\Omega):\ V=\nabla\varphi,\ \ \nabla\cdot V=0,\ \ \varphi\ \ \text{locally constant on } \partial\Omega\},$$

meaning that φ is constant on each component of $\partial\Omega$. This subspace is isomorphic to the absolute homology group $H_2(\Omega;R)$ and also, via Poincaré duality, to the relative homology group $H_1(\Omega,\partial\Omega;R)$, and is hence a finite-dimensional vector space, with dimension equal to the number of components of $\partial\Omega$ minus the number of components of Ω.

The definition of the subspace of *grounded gradients*,

$$\mathrm{GG}=\{V\in \mathrm{VF}(\Omega):V=\nabla\varphi,\ \varphi|_{\partial\Omega}=0\},$$

is self-explanatory.

B. A rough upper bound on helicity

The following result, extracted from Ref. 8, provides a bound on the helicity of any vector field V; this bound depends only on the energy of V and the volume of Ω.

Theorem E: *Let V be a smooth vector field in three-space, defined on the compact domain* Ω *with smooth boundary. Then the helicity* $H(V)$ *of V is bounded by*

$$|H(V)| \leq R(\Omega)\ E(V),$$

where $R(\Omega)$ *is the radius of a round ball having the same volume as* Ω *and* $E(V) = \int_\Omega V \cdot V\, d(\mathrm{vol})$ *is the energy of V.*

This upper bound is not sharp, but it is of the right order of magnitude: for example, the Woltjer spheromak field V on the round ball Ω (shown in Fig. 2) has helicity greater than one-fifth of the asserted upper bound.

Sharp upper bounds obtained by spectral methods will be discussed in the following sections.

C. Properties of the Biot–Savart operators

It is useful to have a clear picture of the image of the modified Biot–Savart operator. We will say that a vector field $V \in \mathrm{K}(\Omega)$ satisfies *Ampere's law* if

$$\int_C V \cdot ds = 0$$

for all closed curves C on $\partial\Omega$ which bound in $R^3 - \Omega$.

We refer the reader to Ref. 10 for proofs of the following three theorems.

Theorem F: *The image of the modified Biot–Savart operator consists of those vector fields* $V \in \mathrm{K}(\Omega)$ *which satisfy Ampere's law.*

Theorem G: *The ordinary and modified Biot–Savart operators* BS *and* BS′ *are bounded operators, and hence they extend to bounded operators on the* L^2 *completions of their domains; there they are both compact and self-adjoint.*

Theorem H: *The equation* $\nabla \times \mathrm{BS}(V) = V$ *holds in* Ω *if and only if* $V \in \mathrm{K}(\Omega)$*, that is, if and only if V is divergence-free and tangent to the boundary of* Ω.

D. Connection with the curl operator

If the vector field V is divergence-free and tangent to the boundary of its domain Ω, then, by Theorem H,

$$\nabla \times \mathrm{BS}(V) = V.$$

Since $\mathrm{BS}(V)$ and $\mathrm{BS}'(V)$ differ by a gradient vector field, we also have

$$\nabla \times \mathrm{BS}'(V) = V.$$

If V is an eigenfield of BS′,

$$\mathrm{BS}'(V) = \lambda V,$$

then

$$\nabla \times V = (1/\lambda) V.$$

Thus the eigenvalue problem for BS′ can be converted to an eigenvalue problem for curl on the image of BS′, which means to a system of partial differential equations. Even though we extended BS′ to the L^2 completion of $\mathrm{K}(\Omega)$ in order to apply the spectral theorem, the eigenfields are smooth vector fields in $\mathrm{K}(\Omega)$; this follows, thanks to elliptic regularity, because on divergence-

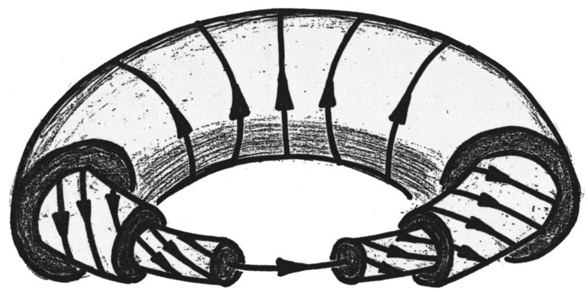

FIG. 1. The Lundquist tokamak field.

free vector fields, the square of the curl is the negative of the Laplacian. Hence these eigenfields can be (and are) discovered by solving this system of partial differential equations (PDEs).

E. Explicit computation of helicity-maximizing vector fields

We solve $\nabla\times V=(1/\lambda)V$ on the flat solid torus $D^2(a)\times S^1$, where $D^2(a)$ is a disk of radius a and S^1 is a circle of any length. Although this is not a subdomain of three-space, the solution here is so clear cut and instructive as to be irresistable; see Ref. 12.

The largest eigenvalue of BS′ on this domain is

$$\lambda(D^2(a)\times S^1)=a/2.405...,$$

where the denominator is the first positive zero of the Bessel function J_0. The corresponding eigenfield, discovered by Lundquist[15] in 1951 in his study of force-free magnetic fields on a cylinder, and known in plasma physics as a *tokamak* field (see Fig. 1), is

$$V=J_1(r/\lambda)\hat{\varphi}+J_0(r/\lambda)\hat{z},$$

expressed in terms of cylindrical coordinates $(r,\ \varphi,\ z)$ and the Bessel functions J_0 and J_1.

It follows that if V is any vector field in $\mathrm{K}(D^2(a)\times S^1)$, then

$$H(V)\leqslant(a/2.405...)E(V),$$

with equality for the above eigenfield V.

We solve $\nabla\times V=(1/\lambda)V$ on the round ball $B^3(a)$ of radius a in terms of spherical Bessel functions in Ref. 13.

The largest eigenvalue of BS′ on this domain is

$$\lambda(B^3(a))=a/4.4934...;$$

the denominator is the first positive zero of

$$(\sin x)/x-\cos x.$$

The corresponding eigenfield is Woltjer's model for the magnetic field in the Crab Nebula,[16] also known in plasma physics as a *spheromak* field (see Fig. 2), and is described below in spherical coordinates $(r,\ \theta,\ \varphi)$ on a ball of radius $a=1$:

$$V(r,\theta,\varphi)=u(r,\theta)\hat{r}+v(r,\theta)\hat{\theta}+w(r,\theta)\hat{\varphi},$$

where

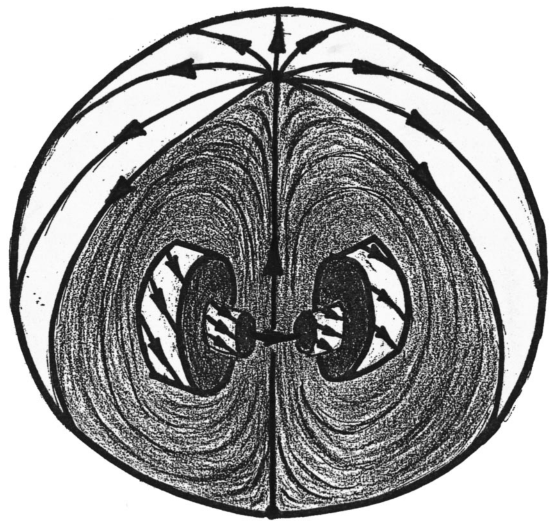

FIG. 2. The Woltjer spheromak field.

$$u(r,\theta) = (2\lambda/r^2)\big((\sin r/\lambda)/(r/\lambda) - \cos r/\lambda\big)\cos\theta,$$

$$v(r,\theta) = (-1/r)\big((\cos r/\lambda)/(r/\lambda) - (\sin r/\lambda)/(r/\lambda)^2 + \sin r/\lambda\big)\sin\theta,$$

$$w(r,\theta) = (1/r)\big((\sin r/\lambda)/(r/\lambda) - \cos r/\lambda\big)\sin\theta.$$

Note that the value $\lambda = 1/4.4934...$ makes both $u(r,\theta)$ and $w(r,\theta)$ vanish when $r = 1$, that is, at the boundary of the ball. As a consequence, the vector field V is tangent to the boundary of the ball, and directed there along the meridians of longitude.

It follows that if V is any vector field in $\mathrm{K}(B^3(a))$, then

$$H(V) \leqslant (a/4.4934...)E(V),$$

with equality for the above eigenfield V.

Compare this with the earlier rough upper bound,

$$|H(V)| \leqslant aE(V),$$

promised by Theorem E.

Comparison of the two pictures above shows how the fundamental features of the helicity-maximizer persist even as the domain changes topological type.

III. THE ISOPERIMETRIC PROBLEM

A. Invariance of helicity

Arnold[4] showed in 1974 that the helicity (he called it the *mean Hopf invariant*) of a vector field V on a closed orientable three-manifold can be defined using just a volume element (rather than a Riemannian metric), provided the vector field is ''homologous to zero.'' To see what he meant by this, convert the vector field V to a two-form ω_V in the usual way by defining $\omega_V(U_1,U_2) = \mathrm{vol}(V,U_1,U_2)$. If V is divergence-free, then ω_V is closed. Arnold called a divergence-free vector field V *homologous to zero* if the corresponding two-form ω_V is exact. If a Riemannian metric compatible with the volume form is present, then a vector field is homologous to zero if and only if it is in the image of curl.

The corresponding results about helicity of vector fields defined on domains in three-space are as follows.

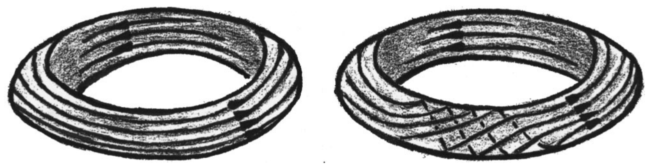

FIG. 3. Helicity can change.

(1) Let Ω_1 be a compact simply-connected domain in three-space with smooth boundary, and V_1 a divergence-free vector field defined on Ω_1 and tangent to its boundary. Let $h:\Omega_1 \rightarrow \Omega_2$ be an orientation-preserving, volume-preserving diffeomorphism, and define $V_2 = h_*(V_1)$. Then the helicity $H(V_1) = H(V_2)$.
(2) The same result holds if we drop the hypothesis that Ω_1 is simply connected, but add the hypothesis that the vector field V_1 is *fluxless* (as defined in the section on the Hodge decomposition theorem).

The arguments are straightforward adaptations of those of Arnold; we do not give them here, nor do we use these two results.

By contrast, if in (1) we drop the hypothesis that Ω_1 is simply connected, and do not replace it with another suitable assumption, then we can have $H(V_1) \neq H(V_2)$ (see Fig. 3).

The invariance property of helicity in three-space that we do need is that it remains constant when the vector field is carried along by a volume-preserving *deformation* of domain, as asserted in Theorem A. We turn to this next.

B. Material derivatives and the transport theorem

Our proof of Theorem A will use *material derivatives* and the *transport theorem* from fluid mechanics, so we pause for a brief reminder, referring the reader to Chap. 1 of Ref. 17 for more details.

Suppose that a fluid is moving through three-space, and that $W(x,t)$ is the velocity of the fluid particle at location x and time t.

Let $F(x,t)$ be some quantity, scalar or vector, defined in the region where the fluid is flowing.

Let DF/Dt denote the rate of change of F as measured by a person moving with the flow. This quantity is known as the *material derivative* of F and is given by

$$
\begin{aligned}
DF/Dt &= \partial F/\partial t + \sum_i (\partial F/\partial x_i)(dx_i/dt) \\
&= \partial F/\partial t + (W\cdot\nabla)F.
\end{aligned}
$$

Let $d(\mathrm{vol})$ be a small chunk of fluid moving with the flow. Then

$$(D/Dt)d(\mathrm{vol}) = (\nabla\cdot W)d(\mathrm{vol}).$$

Hence

$$
\begin{aligned}
(D/Dt)(F d(\mathrm{vol})) &= (DF/Dt)d(\mathrm{vol}) + F(D/Dt)d(\mathrm{vol}) \\
&= (\partial F/\partial t + (W\cdot\nabla)F + F(\nabla\cdot W))d(\mathrm{vol}).
\end{aligned}
$$

Suppose that Ω_t is a region moving with the fluid and always containing the same fluid particles. Then the *transport theorem* asserts that

$$(d/dt)\int_{\Omega_t} F(x,t)d(\text{vol}) = \int_{\Omega_t} (D/Dt)(F(x,t)d(\text{vol}))$$

$$= \int_{\Omega_t} (\partial F/\partial t + (W\cdot\nabla)F + F(\nabla\cdot W))d(\text{vol}).$$

If the fluid is incompressible (that is, the flow is volume-preserving), then W is divergence-free and the last term in the integrand above is zero.

C. Proof of Theorem A

Let Ω be a compact domain with smooth boundary in three-space, and V a divergence-free vector field defined on Ω and tangent to its boundary.

Let $h_t:\Omega\to\Omega_t$ be a smooth family of volume-preserving diffeomorphisms of Ω into R^3, with h_0 the identity.

Define a vector field W on Ω by $W(x) = d/dt|_{t=0}h_t(x)$. This vector field records the initial velocity of the deformation h_t. Since each h_t is volume preserving, W is divergence-free.

Let $V_t = (h_t)_* V$, a smooth divergence-free vector field defined on Ω_t and tangent to its boundary. Thus V_t is frozen into Ω_t as it deforms.

Theorem A asserts that the helicity $H(V_t)$ is independent of t.

We will demonstrate this by showing that the derivative $(d/dt)H(V_t)$ is zero, and since the argument will be independent of which instant of time we are at, it will be sufficient to show that

$$d/dt|_{t=0}H(V_t) = 0.$$

We begin by writing

$$H(V_t) = \int_{\Omega_t} \text{BS}(V_t)\cdot V_t\, d(\text{vol}),$$

and then differentiate with respect to t at $t=0$:

$$d/dt|_{t=0}H(V_t) = d/dt|_{t=0}\int_{\Omega_t} \text{BS}(V_t)\cdot V_t\, d(\text{vol}),$$

$$= \int_{\Omega} D/Dt|_{t=0}(\text{BS}(V_t)\cdot V_t d(\text{vol}))$$

$$= \int_{\Omega} (D/Dt)|_{t=0}(\text{BS}(V_t)\cdot V_t))d(\text{vol}),$$

where the next-to-last equality uses the material derivative D/Dt and the transport theorem, as reviewed in the previous section, while the last equality uses the fact that the diffeomorphisms h_t are volume preserving.

Now

$$D/Dt|_{t=0}(\text{BS}(V_t)\cdot V_t) = (D/Dt|_{t=0}\text{BS}(V_t))\cdot V + \text{BS}(V)\cdot(D/Dt|_{t=0}V_t).$$

From the previous section we have

$$D/Dt|_{t=0}V_t = d/dt|_{t=0}V_t + (W\cdot\nabla)V,$$

$$D/Dt|_{t=0}\text{BS}(V_t) = d/dt|_{t=0}\text{BS}(V_t) + (W\cdot\nabla)\text{BS}(V),$$

indulging our habit of writing d/dt in place of $\partial/\partial t$.

We need to learn the values of $d/dt|_{t=0}V_t$ and $d/dt|_{t=0}\mathrm{BS}(V_t)$.

We begin by staying put at the fixed location x in the interior of Ω and watching the vector $V_t(x)$ change with time:

$$\begin{aligned}(d/dt)|_{t=0}V_t(x)&=\lim_{t\to 0}(1/t)(V_t(x)-V(x))\\&=\lim_{t\to 0}(1/t)((h_t)_*V(h_t^{-1}x)-V(x))\\&=[V,W](x),\end{aligned}$$

the value at x of the Lie bracket $[V,W]$ of the vector fields V and W.

Remaining at x, we watch the magnetic field $\mathrm{BS}(V_t(x))$ change with time. This change is due to two influences: for one thing, the vector field V_t is changing; for another, the domain Ω_t is shifting.

The contribution to $d/dt|_{t=0}\mathrm{BS}(V_t)$ due to the changing vector field is simply $\mathrm{BS}(d/dt|_{t=0}V_t)$ by the linearity of the operator BS.

The contribution to $d/dt|_{t=0}\mathrm{BS}(V_t)$ due to the shifting domain is the magnetic field due to a surface current distribution $(W\cdot n)V$ along the boundary of Ω. We will record this contribution as

$$\mathrm{BS}((W\cdot n)V|_{\partial\Omega}).$$

Thus

$$\begin{aligned}d/dt|_{t=0}\mathrm{BS}(V_t)&=\mathrm{BS}(d/dt|_{t=0}V)+\mathrm{BS}((W\cdot n)V|_{\partial\Omega})\\&=\mathrm{BS}([V,W])+\mathrm{BS}((W\cdot n)V|_{\partial\Omega}).\end{aligned}$$

Having learned the values of $d/dt|_{t=0}V_t$ and $d/dt|_{t=0}\mathrm{BS}(V_t)$, we get the following formulas for the material derivatives of V_t and $\mathrm{BS}(V_t)$:

$$D/Dt|_{t=0}V_t=[V,W]+(W\cdot\nabla)V,$$

$$D/Dt|_{t=0}\mathrm{BS}(V_t)=\mathrm{BS}([V,W])+\mathrm{BS}((W\cdot n)V|_{\partial\Omega})+(W\cdot\nabla)\mathrm{BS}(V).$$

We insert this information into our computation of the time rate of change of helicity:

$$\begin{aligned}d/dt|_{t=0}H(V_t)&=\int_\Omega D/Dt|_{t=0}(\mathrm{BS}(V_t)\cdot V_t))d(\mathrm{vol})\\&=\int_\Omega(D/Dt|_{t=0}\mathrm{BS}(V_t)\cdot V+\mathrm{BS}(V)\cdot(D/Dt|_{t=0}V_t)d(\mathrm{vol})\\&=\langle(D/Dt|_{t=0}\mathrm{BS}(V_t)),V\rangle+\langle\mathrm{BS}(V),(D/Dt|_{t=0}V_t)\rangle\\&=\langle\mathrm{BS}([V,W])+\mathrm{BS}((W\cdot n)V|_{\partial\Omega})+(W\cdot\nabla)\mathrm{BS}(V),V\rangle+\langle\mathrm{BS}(V),[V,W]+(W\cdot\nabla)V\rangle\\&=\langle\mathrm{BS}([V,W]),V\rangle+\langle\mathrm{BS}(V),[V,W]\rangle+\langle\mathrm{BS}((W\cdot n)V|_{\partial\Omega}),V\rangle+\langle(W\cdot\nabla)\mathrm{BS}(V),V\rangle\\&\quad+\langle\mathrm{BS}(V),(W\cdot\nabla)V\rangle,\end{aligned}$$

where we have reordered the five terms for the convenience of further computation.

To begin, the first two terms are equal, thanks to the self-adjointness of the operator BS, and we combine them as $2\langle\mathrm{BS}(V),[V,W]\rangle$.

The last two terms combine to yield

$$\int_\Omega W\cdot\nabla(\mathrm{BS}(V)\cdot V)d(\mathrm{vol}).$$

The middle term can be rewritten as

$$\langle \mathrm{BS}((W\cdot n)V|_{\partial\Omega}),V\rangle = \int_\Omega \mathrm{BS}((W\cdot n)V|_{\partial\Omega})\cdot V\, d(\mathrm{vol})$$
$$= \int_{\partial\Omega} (W\cdot n)V\cdot \mathrm{BS}(V)\, d(\mathrm{area}),$$

by using a version of the symmetry of BS appropriate to this situation.

Assembling, we get

$$d/dt|_{t=0} H(V_t) = 2\langle \mathrm{BS}(V),[V,W]\rangle + \int_{\partial\Omega} (W\cdot n)V\cdot\mathrm{BS}(V)\, d(\mathrm{area}) + \int_\Omega W\cdot\nabla(\mathrm{BS}(V)\cdot V)\, d(\mathrm{vol}).$$

Our job is now to process the three terms on the right-hand side of this equation and show that they add up to zero.

We begin with the first term.

Recall the formula

$$\nabla\times(A\times B) = (B\cdot\nabla)A - (A\cdot\nabla)B + A(\nabla\cdot B) - B(\nabla\cdot A)$$
$$= [B, A] + A(\nabla\cdot B) - B(\nabla\cdot A).$$

Apply this formula with $A=W$ and $B=V$, keeping in mind that both V and W are divergence-free, to get

$$\nabla\times(W\times V) = [V,W].$$

Next, recall the formula

$$\nabla\cdot(A\times B) = B\cdot(\nabla\times A) - A\cdot(\nabla\times B).$$

Apply this formula with $A=W\times V$ and $B=\mathrm{BS}(V)$ to get

$$\nabla\cdot((W\times V)\times\mathrm{BS}(V)) = \mathrm{BS}(V)\cdot(\nabla\times(W\times V)),$$

since the term

$$(W\times V)\cdot(\nabla\times\mathrm{BS}(V)) = (W\times V)\cdot V = 0,$$

because V is divergence-free and tangent to the boundary of Ω.

Thus

$$\langle \mathrm{BS}(V),[V,W]\rangle = \langle \mathrm{BS}(V),\nabla\times(W\times V)\rangle$$
$$= \int_\Omega \mathrm{BS}(V)\cdot(\nabla\times(W\times V))\, d(\mathrm{vol})$$
$$= \int_\Omega \nabla\cdot((W\times V)\times\mathrm{BS}(V))\, d(\mathrm{vol})$$
$$= \int_{\partial\Omega} ((W\times V)\times\mathrm{BS}(V))\cdot n\, d(\mathrm{area})$$
$$= \int_{\partial\Omega} (\mathrm{BS}(V)\times(V\times W))\cdot n\, d(\mathrm{area})$$

$$= \int_{\partial\Omega} ((\mathrm{BS}(V)\cdot W)V - (\mathrm{BS}(V)\cdot V)W)\cdot n \, d(\text{area})$$

$$= -\int_{\partial\Omega} (\mathrm{BS}(V)\cdot V)(W\cdot n)\, d(\text{area}),$$

since V is tangent to $\partial\Omega$.

The middle term in our expression for $d/dt|_{t=0}H(V_t)$,

$$\int_{\partial\Omega} (W\cdot n)V\cdot \mathrm{BS}(V)\, d(\text{area}) = \int_{\partial\Omega} (\mathrm{BS}(V)\cdot V)(W\cdot n)\, d(\text{area}),$$

needs no further modification.

We process the final term as follows.

$$\int_{\Omega} W\cdot\nabla(\mathrm{BS}(V)\cdot V)\, d(\text{vol}) = \int_{\Omega} \nabla\cdot((\mathrm{BS}(V)\cdot V)W)\, d(\text{vol}),$$

since W is divergence-free, and then

$$= \int_{\partial\Omega} (\mathrm{BS}(V)\cdot V)(W\cdot n)\, d(\text{area}).$$

Putting this all together, we get

$$d/dt|_{t=0}H(V_t) = 2\langle \mathrm{BS}(V),[V,W]\rangle + \int_{\partial\Omega} (W\cdot n)V\cdot \mathrm{BS}(V)\, d(\text{area})$$

$$+ \int_{\Omega} W\cdot\nabla(\mathrm{BS}(V)\cdot V)\, d(\text{vol}).$$

$$= -2\int_{\partial\Omega} (\mathrm{BS}(V)\cdot V)(W\cdot n)\, d(\text{area}) + \int_{\partial\Omega} (\mathrm{BS}(V)\cdot V)(W\cdot n)\, d(\text{area})$$

$$+ \int_{\partial\Omega} (\mathrm{BS}(V)\cdot V)(W\cdot n)\, d(\text{area}) = 0,$$

completing the proof of Theorem A.

D. A general first variation formula for helicity

We continue to assume that the vector field V is divergence-free and tangent to the boundary of Ω, but for this section only we give up the assumption that the deformation $h_t:\Omega\to\Omega_t$ is volume preserving, and hence lose the condition that W is divergence-free.

As a result, the helicity $H(V_t)$ will no longer be independent of t; instead, we will derive a first variation formula for helicity involving the term

$$\int_{\Omega} (\mathrm{BS}(V)\cdot V)(\nabla\cdot W)\, d(\text{vol}).$$

To get this formula, we simply modify the proof of Theorem A at the three locations where we used the old hypothesis that $\nabla\cdot W=0$, as follows.

First, at the beginning of the proof, when we apply the transport theorem, we must now add the above integral into our formula for the first variation of helicity.

Again in the middle, where we process the term $2\langle \mathrm{BS}(V),[V,W]\rangle$, we must now use the identity

$$[V,W]=\nabla\times(W\times V)+V(\nabla\cdot W),$$

and so gain the term $2\langle \mathrm{BS}(V),V(\nabla\cdot W)\rangle$, which is twice the above integral.

And finally at the end, when we process the term $\int_\Omega W\cdot\nabla(\mathrm{BS}(V)\cdot V)\,d(\mathrm{vol})$, we must now use the identity

$$W\cdot\nabla(\mathrm{BS}(V)\cdot V)=\nabla\cdot((\mathrm{BS}(V)\cdot V)W)-(\mathrm{BS}(V)\cdot V)(\nabla\cdot W),$$

and therefore must subtract our new integral from the formula.

The net result, $1+2-1=2$, is that we must now add twice our new integral to the old first variation formula for helicity. Since helicity was invariant under volume-preserving deformations, the new formula reads

$$\delta H(V)=2\int_\Omega(\mathrm{BS}(V)\cdot V)(\nabla\cdot W)\,d(\mathrm{vol}).$$

We can do a spot check on this new formula, as follows.

Let h_t be a gradual expansion of all of three-space defined by $h_t(x)=(1+t)x$, and then restrict h_t to the domain Ω. The initial velocity field W of this deformation is the position vector

$$\mathbf{r}=x\hat{x}+y\hat{y}+z\hat{z},$$

and hence $\nabla\cdot W=3$. The vector field V_t on $\Omega_t=(1+t)\Omega$ is defined by the formula $V_t((1+t)x)=(1+t)V(x)$.

But then in the helicity formula

$$H(V)=(1/4\pi)\int_{\Omega\times\Omega}V(x)\times V(y)\cdot(x-y)/|x-y|^3\,d(\mathrm{vol}_x)\,d(\mathrm{vol}_y),$$

each term in the integrand, including $d(\mathrm{vol}_x)$ and $d(\mathrm{vol}_y)$, will be multiplied by an appropriate power of $(1+t)$ when computing $H(V_t)$, with the net result

$$H(V_t)=(1+t)^6H(V).$$

It follows that

$$\delta H(V)=(d/dt)H(V_t)|_{t=0}=6H(V).$$

Since $\nabla\cdot W=3$, we get the same result from our new formula,

$$\begin{aligned}\delta H(V)&=2\int_\Omega(\mathrm{BS}(V)\cdot V)(\nabla\cdot W)\,d(\mathrm{vol}),\\&=6\int_\Omega \mathrm{BS}(V)\cdot V\,d(\mathrm{vol})\\&=6H(V),\end{aligned}$$

providing a morsel of confirmation.

E. Variation of energy

As usual, Ω is a compact domain in three-space with smooth boundary.

But now we let V be *any* smooth vector field on Ω, thus an arbitrary member of VF(Ω). We do *not* assume that V is divergence-free, and we do *not* assume that V is tangent to the boundary of Ω.

Our first variation of energy formula will be presented in a way which makes clear the simplifying effects of the various special assumptions about V.

Let $h_t:\Omega\rightarrow\Omega_t$ be a smooth one-parameter family of volume-preserving diffeomorphisms of Ω into three-space. As before, we define the vector field W on Ω by $W(x)=d/dt|_{t=0}h_t(x)$. Since the deformation is volume preserving, W is divergence-free.

Again we let our original vector field V on Ω be carried along by the deformation, and so define the vector field V_t on Ω_t by the formula $V_t=(h_t)_*V$.

This time we consider the energy of the vector field V_t on Ω_t,

$$E(V_t)=\int_{\Omega_t} V_t\cdot V_t\, d(\mathrm{vol}),$$

and seek a useful formula for its first variation,

$$\delta E(V)=d/dt|_{t=0}E(V_t).$$

Theorem I:

$$\delta E(V)=2\langle V\times(\nabla\times V),W\rangle-2\langle(\nabla\cdot V)V,W\rangle$$
$$+2\int_{\partial\Omega}(V\cdot W)(V\cdot n)\, d(\mathrm{area})-\int_{\partial\Omega}|V|^2(W\cdot n)\, d(\mathrm{area}).$$

F. Proof of Theorem I

Consider the first variation of energy,

$$\delta E(V)=d/dt|_{t=0}E(V_t)=d/dt|_{t=0}\int_{\Omega_t}V_t\cdot V_t\, d(\mathrm{vol})$$
$$=\int_{\Omega}D/Dt|_{t=0}(V_t\cdot V_t)\, d(\mathrm{vol}),$$

since the diffeomorphisms h_t are volume preserving.

Continuing, we get

$$\int_{\Omega}D/Dt|_{t=0}(V_t\cdot V_t)\, d(\mathrm{vol})=2\int_{\Omega}V\cdot DV_t/Dt|_{t=0}\, d(\mathrm{vol})$$
$$=2\int_{\Omega}V\cdot([V,W]+(W\cdot\nabla)V)\, d(\mathrm{vol})$$
$$=2\int_{\Omega}V\cdot[V,W]\, d(\mathrm{vol})+\int_{\Omega}W\cdot\nabla|V|^2\, d(\mathrm{vol}).$$

The first integral on the right is simply the L^2 inner product $2\langle V,[V,W]\rangle$.

The second integral on the right can be written as

$$\int_{\Omega}\nabla\cdot(|V|^2W)\, d(\mathrm{vol}),$$

since W is divergence-free, and then as

$$\int_{\partial\Omega} |V|^2 W \cdot n \, d(\text{area})$$

by the divergence theorem.

So we have shown that

$$\delta E(V) = 2\langle V, [V,W]\rangle + \int_{\partial\Omega} |V|^2 W \cdot n \, d(\text{area}).$$

This formula can be regarded as a way station on route to our final answer. It is useful in itself if $[V,W]=0$, which means that $V_t(x)$ agrees with $V(x)$ to first order at $t=0$. In that case

$$\delta E(V) = \int_{\partial\Omega} |V|^2 W \cdot n \, d(\text{area}),$$

which, upon a moment's reflection, is intuitively plausible.

However, in general we will do better to further process the term $\langle V, [V,W]\rangle$.

Our handling of the term $\langle V, [V,W]\rangle$ here will be very similar to our treatment of the term $\langle \mathrm{BS}(V), [V,W]\rangle$ in the proof of Theorem A.

Once again we use from vector calculus the formula

$$\nabla \times (A \times B) = (B \cdot \nabla)A - (A \cdot \nabla)B + A(\nabla \cdot B) - B(\nabla \cdot A) = [B,A] + A(\nabla \cdot B) - B(\nabla \cdot A),$$

again with $A=W$ and $B=V$, but this time we only know that $\nabla \cdot W = 0$.

We get

$$\nabla \times (W \times V) = [V,W] + W(\nabla \cdot V),$$

or

$$[V,W] = \nabla \times (W \times V) - W(\nabla \cdot V).$$

Thus

$$\langle V, [V,W]\rangle = \langle V, \nabla \times (W \times V)\rangle - \langle V, W(\nabla \cdot V)\rangle.$$

Focus on the first term on the right, and let us try to take the curl operator away from $(W \times V)$ and give it to V. To this end, we once again recall the formula:

$$\nabla \cdot (A \times B) = (\nabla \times A) \cdot B - A \cdot (\nabla \times B).$$

This time we apply the formula with $A=V$ and $B=W \times V$ to get

$$\nabla \cdot (V \times (W \times V)) = (\nabla \times V) \cdot (W \times V) - V \cdot (\nabla \times (W \times V)),$$

or

$$V \cdot (\nabla \times (W \times V)) = (\nabla \times V) \cdot (W \times V) + \nabla \cdot (V \times (V \times W)).$$

Now integrate this last formula over Ω and apply the divergence theorem to get

$$\begin{aligned}\langle V, \nabla \times (W \times V)\rangle &= \langle \nabla \times V, W \times V\rangle + \int_{\partial\Omega} (V \times (V \times W)) \cdot n \, d(\text{area}) \\ &= \langle V \times (\nabla \times V), W\rangle + \int_{\partial\Omega} (V \cdot W)(V \cdot n) \, d(\text{area}) - \int_{\partial\Omega} |V|^2 W \cdot n \, d(\text{area}),\end{aligned}$$

where the last equality relies on the identity

$$V\times(V\times W)=(V\cdot W)V-(V\cdot V)W.$$

Finally, we get

$$\begin{aligned}\delta E(V)&=2\langle V,[V,W]\rangle+\int_{\partial\Omega}|V|^2 W\cdot n\,d(\text{area})\\&=2\langle V,\nabla\times(W\times V)\rangle-2\langle V,W(\nabla\cdot V)\rangle+\int_{\partial\Omega}|V|^2 W\cdot n\,d(\text{area})\\&=2\langle V\times(\nabla\times V),W\rangle+2\int_{\partial\Omega}(V\cdot W)(V\cdot n)\,d(\text{area})-2\int_{\partial\Omega}|V|^2 W\cdot n\,d(\text{area})\\&\quad-2\langle V,W(\nabla\cdot V)\rangle+\int_{\partial\Omega}|V|^2 W\cdot n\,d(\text{area})\\&=2\langle V\times(\nabla\times V),W\rangle-2\langle(\nabla\cdot V)V,W\rangle+2\int_{\partial\Omega}(V\cdot W)(V\cdot n)\,d(\text{area})\\&\quad-\int_{\partial\Omega}|V|^2 W\cdot n\,d(\text{area}),\end{aligned}$$

completing the proof of Theorem I.

G. Proof of Theorem B and other corollaries to Theorem I

Consider once again the first variation of energy formula given by Theorem I:

$$\begin{aligned}\delta E(V)=&\,2\langle V\times(\nabla\times V),W\rangle-2\langle(\nabla\cdot V)V,W\rangle\\&+2\int_{\partial\Omega}(V\cdot W)(V\cdot n)\,d(\text{area})-\int_{\partial\Omega}|V|^2 W\cdot n\,d(\text{area}).\end{aligned}$$

If V is divergence-free, then the second term on the right vanishes; if V is tangent to the boundary of Ω, then the third term vanishes.

We are left with

$$\delta E(V)=2\langle V\times(\nabla\times V),W\rangle-\int_{\partial\Omega}|V|^2(W\cdot n)\,d(\text{area}),$$

which is exactly the assertion of Theorem B.

We turn now to a sequence of corollaries to Theorem I.

Corollary 1: *If V is divergence-free and tangent to the boundary of its domain Ω, then*

$$\delta E(V)=-2\langle(V\cdot\nabla)V,W\rangle.$$

Proof: We begin with the formula of Theorem B, and make the substitution

$$\int_{\partial\Omega}|V|^2 W\cdot n\,d(\text{area})=\int_{\Omega}\nabla\cdot(|V|^2 W)\,d(\text{vol})=\int_{\Omega}(\nabla|V|^2)\cdot W\,d(\text{vol}),$$

since W is divergence-free, to get

$$\delta E(V)=\langle 2V\times(\nabla\times V)-\nabla|V|^2,W\rangle.$$

Then we use the formula

$$\nabla(A\cdot B)=A\times(\nabla\times B)+B\times(\nabla\times A)+(A\cdot\nabla)B+(B\cdot\nabla)A,$$

with $A=B=V$ to get

$$\nabla|V|^2=2V\times(\nabla\times V)+2(V\cdot\nabla)V,$$

from which the desired result follows.

Corollary 2: *If V is divergence-free and tangent to the boundary of* Ω, *then* $\delta E(V)=0$ *for all divergence-free W if and only if* $(V\cdot\nabla)V$ *is the gradient of a function which vanishes on* $\partial\Omega$.

Proof: Recall, from the Hodge decomposition theorem, that the vector fields on Ω which are gradients of functions vanishing on $\partial\Omega$ form the subspace GG of grounded gradients, which is the orthogonal complement inside $VF(\Omega)$ of the subspace of divergence-free vector fields.

Then Corollary 2 follows immediately from Corollary 1.

Corollary 3: *Let V be divergence-free and tangent to* $\partial\Omega$, *and suppose that* $\delta E(V)=0$ *for all divergence-free W. Then on* $\partial\Omega$, *the orbits of V are constant speed geodesics.*

Caution: That constant speed may, at least in principle, vary from geodesic to geodesic.

Proof: Let $g_t(p)$ be the orbit of V which at time 0 passes through the point p. Thus $(d/dt)g_t(p)=V(g_t(p))$.

A straightforward computation shows that the acceleration along this orbit is given by

$$(d^2/dt^2)g_t(p)=((V\cdot\nabla)V)(g_t(p)).$$

Now the hypotheses on V imply, by Corollary 2, that $(V\cdot\nabla)V$ is the gradient of a function that vanishes on $\partial\Omega$, and hence that $(V\cdot\nabla)V$ is orthogonal to $\partial\Omega$.

Thus if p, and hence the orbit $g_t(p)$ through it, lie on $\partial\Omega$, then the acceleration vector $(d^2/dt^2)g_t(p)$ is orthogonal to $\partial\Omega$, and therefore this orbit is a constant speed geodesic on $\partial\Omega$.

Corollary 4: *If V is divergence-free and tangent to the boundary of* Ω, *and is an eigenfield of the curl operator, then*

$$\delta E(V)=-\int_{\partial\Omega}|V|^2(W\cdot n)\,d(\text{area}).$$

Proof: This follows immediately from the first variation formula for the energy given in Theorem I, since the hypotheses on V imply that the first three terms on the right-hand side of the formula vanish.

Note that if the vector field V is an eigenfield of the modified Biot–Savart operator BS′, then it is also an eigenfield of curl, according to Theorem H, and hence the above formula holds.

In particular, this formula holds when the vector field V maximizes helicity for given energy.

Corollary 5: *If V is divergence-free and tangent to the boundary of* Ω, *and is an eigenfield of the curl operator, then* $\delta E(V)=0$ *for all volume-preserving deformations of* Ω *if and only if* $|V|$ *is constant on the boundary of each component of* Ω.

Proof: We begin with the formula of Corollary 4 for $\delta E(V)$ under these circumstances:

$$\delta E(V)=-\int_{\partial\Omega}|V|^2(W\cdot n)\,d(\text{area}).$$

Since W is divergence-free, we have $\int_{\partial\Omega_i}W\cdot n\,d(\text{area})=0$ for each component Ω_i of Ω. So if $|V|$ is constant on each $\partial\Omega_i$, we get

$$\delta E(V)=-\int_{\partial\Omega}|V|^2(W\cdot n)\,d(\text{area})=-\sum_i\int_{\partial\Omega_i}|V|^2(W\cdot n)\,d(\text{area})$$
$$=-\sum_i|V|^2\int_{\partial\Omega_i}W\cdot n\,d(\text{area})=0.$$

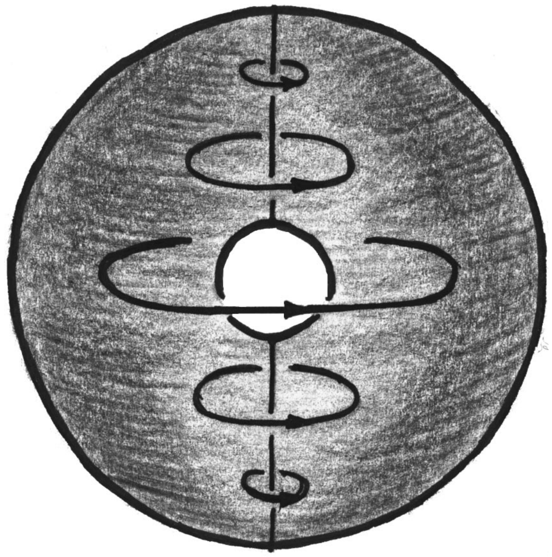

FIG. 4. An example of variation of energy.

If $|V|$ is not constant on the boundary of the component Ω_i of Ω, pick two points p and q on $\partial\Omega_i$ where $|V(p)| \neq |V(q)|$. Connect p and q by a thin tube running through Ω_i. Then define a volume-preserving deformation $h_t : \Omega \rightarrow \Omega_t$ which is entirely supported on this thin tube, pushing the material in it so that it dimples in from $\partial\Omega_i$ near p and bulges out near q. The corresponding initial velocity vector field $W(x) = d/dt|_{t=0} h_t(x)$ is also supported in this tube, and satisfies $(W \cdot n) \leqslant 0$ on $\partial\Omega_i$ near p and $(W \cdot n) \geqslant 0$ on $\partial\Omega_i$ near q, and of course $\int_{\partial\Omega_i} W \cdot n \, d(\text{area}) = 0$. Then clearly

$$\delta E(V) = -\int_{\partial\Omega} |V|^2 (W \cdot n) \, d(\text{area}) = -\int_{\partial\Omega_i} |V|^2 (W \cdot n) \, d(\text{area}) \neq 0,$$

completing the proof of the corollary.

H. Variation of energy—an illustrative example

The domain Ω in this example is the spherical shell centered at the origin in three-space, with boundary spheres of radii $a < b$ (see Fig. 4).

The vector field V, given in spherical coordinates by

$$V = r \sin\theta \; \hat{\varphi},$$

is the velocity field of rigid rotation of R^3 about the z axis, and is divergence-free and tangent to the boundary of Ω.

The vector field

$$W = (1/r^2)\hat{r},$$

defined on $R^3 -$ origin, is divergence-free and is the infinitesimal generator of the one-parameter group $\{h_t\}$ of volume-preserving diffeomorphisms of $R^3 -$ origin, given by

$$h_t(r, \theta, \varphi) = ((r^3 + 3t)^{1/3}, \theta, \varphi).$$

The vector field V is invariant under the flow $\{h_t\}$, that is, $(h_t)_* V = V$.

The energy

$$E(V)=\int_\Omega |V|^2\, d(\mathrm{vol})$$

of V inside Ω can be computed by straightforward integration, and has the value

$$E(V)=(8\pi/15)(b^5-a^5).$$

Let $\Omega_t=h_t(\Omega)$ and $V_t=(h_t)_* V=V$. The energy E_t of V_t inside Ω_t is given by

$$E(V_t)=(8\pi/15)((b^3+3t)^{5/3}-(a^3+3t)^{5/3}),$$

and hence

$$\delta E(V)=(d/dt)|_{t=0}E_t=(8\pi/3)(b^2-a^2).$$

Now consider the formula

$$\delta E(V)=2\langle V,\ [V,W]\rangle+\int_{\partial\Omega}|V|^2(W\cdot n)\, d(\mathrm{area}),$$

obtained during the proof of Theorem I. In the present example, $[V,W]=0$, so the formula simplifies to

$$\delta E(V)=\int_{\partial\Omega}V^2(W\cdot n)\, d(\mathrm{area}).$$

The right-hand side can be computed by direct integration, yielding $(8\pi/3)(b^2-a^2)$, which coincides with the value obtained above by computing the left-hand side directly.

Now consider the formula

$$\delta E(V)=2\langle V\times(\nabla\times V),W\rangle-\int_{\partial\Omega}|V|^2(W\cdot n)\, d(\mathrm{area}),$$

from Theorem B.

Direct computation shows that the first term on the right-hand side is $(16\pi/3)(b^2-a^2)$, thus providing yet another confirmation.

Proof of Theorem C

Recall the setup. Ω is a compact domain with smooth boundary in three-space. V is a divergence-free vector field defined on Ω and tangent to its boundary. $h_t:\Omega\to\Omega_t$ is a smooth volume-preserving deformation of Ω, with h_0 the identity. W is the vector field on Ω defined by $W(x)=d/dt|_{t=0}h_t(x)$.

We are seeking a first variation formula for the Rayleigh quotient

$$\lambda(V)=H(V)/E(V),$$

that is to say, a formula for

$$\delta\lambda(V)=d/dt|_{t=0}\lambda(V_t).$$

The first part of Theorem C asserts that

$$\delta\lambda(V)=\lambda(V)\,\frac{-2\langle V\times(\nabla\times V),W\rangle+\int_{\partial\Omega}|V|^2(W\cdot n)\, d(\mathrm{area})}{\int_\Omega |V|^2\, d(\mathrm{vol})}.$$

This is an easy consequence of Theorems A and B, as follows.

According to Theorem A,

$$\delta H(V)=0.$$

Hence

$$\begin{aligned}\delta\lambda(V)&=\delta(H(V)/E(V))\\&=(-H(V)/E(V)^2)\,\delta E(V)\\&=(H(V)/E(V))(-\delta E(V)/E(V))\\&=\lambda(V)(-\delta E(V)/E(V)).\end{aligned}$$

Substituting the value

$$\delta E(V)=2\langle V\times(\nabla\times V),W\rangle-\int_{\partial\Omega}|V|^2(W\cdot n)\,d(\text{area})$$

from Theorem B, and the definition

$$E(V)=\int_{\Omega}|V|^2\,d(\text{vol}),$$

we get the desired formula for $\delta\lambda(V)$.

The second part of Theorem C asserts that if V is an eigenfield of the modified Biot–Savart operator BS′, say $\text{BS}'(V)=\lambda(V)V$, then

$$\delta\lambda(V)=\lambda(V)\,\frac{\int_{\partial\Omega}|V|^2(W\cdot n)\,d(\text{area})}{\int_{\Omega}|V|^2\,d(\text{vol})}.$$

We saw earlier that curl is a left inverse to BS′. Hence $\nabla\times V=\lambda(V)^{-1}V$. Thus we have $V\times(\nabla\times V)=0$, and so the second part of Theorem C follows from the first.

The third part of Theorem C asserts that if this eigenfield V corresponds to the largest eigenvalue $\lambda(\Omega)$ of BS′ on Ω, then

$$\delta\lambda(\Omega)\geqslant\lambda(\Omega)\,\frac{\int_{\partial\Omega}|V|^2(W\cdot n)\,d(\text{area})}{\int_{\Omega}|V|^2\,d(\text{vol})}.$$

In this case, $\lambda(\Omega)=\lambda(V)$. At the same time, $\lambda(\Omega_t)\geqslant\lambda(V_t)$. We signal this by writing $\delta\lambda(\Omega)\geqslant\delta\lambda(V)$, without meaning to suggest that $\lambda(\Omega_t)$ depends differentiably on t. Thus the third part of Theorem C follows from the second.

Theorem C is proved.

F. An alternative version of Theorem C

As mentioned in the Introduction, the largest eigenvalue of a smooth one-parameter family of self-adjoint matrices does not always vary smoothly, and, as a result of this, our first variation formula for the largest eigenvalue $\lambda(\Omega)$ of the modified Biot–Savart operator BS′ appears as an inequality rather than an equality:

$$\delta\lambda(\Omega)\geqslant\lambda(\Omega)\,\frac{\int_{\partial\Omega}|V|^2(W\cdot n)\,d(\text{area})}{\int_{\Omega}|V|^2\,d(\text{vol})}.$$

In this section we will describe an alternative first variation formula which appears as an equality.

We begin with a compact domain Ω with smooth boundary in three-space, and a smooth volume-preserving deformation $h_t:\Omega\rightarrow\Omega_t$, which depends *analytically on t*, with h_0 the identity. We are interested in the eigenvalues and eigenfields of the modified Biot–Savart operators

$$\mathrm{BS}'_t:\ \mathbf{K}(\Omega_t)\rightarrow\mathbf{K}(\Omega_t).$$

Recall that the eigenfields lie in $\mathbf{K}(\Omega_t)$, rather than in its L^2 completion, as a consequence of elliptic regularity.

Consider a single eigenvalue λ of $\mathrm{BS}'=\mathrm{BS}'_0$, of finite multiplicity m. We assume that I is an interval of real numbers containing the eigenvalue λ and no other eigenvalues of BS'. Then the Rellich perturbation theorem[18] can be used to show that for t sufficiently small, there exist m real-valued functions $\lambda_1(t),\ldots,\lambda_m(t)$, each taking the value λ when $t=0$, and each depending analytically on t, such that the portion of the spectrum of BS'_t which lies within the interval I consists of just these eigenvalues, with total multiplicity m. Moreover, the theorem promises that there are m vector fields $V_1(t),\ldots,V_m(t)$ in $K(\Omega_t)$, each depending analytically on t, which form a corresponding orthonormal system of eigenfields.

Now let $\lambda_i(t)$ and $V_i(t)$ be one of the above eigenvalue functions and its corresponding eigenfield function. We have $\lambda=\lambda_i(0)$; for simplicity of notation, we will write $V=V_i(0)$, and also $\delta\lambda=d\lambda_i(t)/dt|_{t=0}$. As usual, W will denote the initial velocity vector field of the deformation h_t.

Then the following first variation equality holds:

$$\delta\lambda=\lambda\,\frac{\int_{\partial\Omega}|V|^2(W\cdot n)\,d(\mathrm{area})}{\int_{\Omega}|V|^2\,d(\mathrm{vol})}.$$

We left the denominator in place on the right-hand side to cover the case when V does not have L^2 norm equal to 1.

We compare this first variation formula with that appearing in Theorem C:

(1) The above formula is an equality, while its counterpart in Theorem C is an inequality.
(2) The above formula requires the smooth deformation of domain to be analytic in the time parameter t, unlike its counterpart in Theorem C. Indeed, the Rellich perturbation theorem is false when the family of operators is only C^∞ in t.
(3) The above formula holds for all eigenvalues λ of BS', but only for the eigenvalue functions promised by the Rellich theorem; in particular, the largest eigenvalue function $\lambda(\Omega_t)$ may not be analytic in t. By contrast, the corresponding formula in Theorem C holds for the largest eigenvalue function $\lambda(\Omega_t)$.
(4) The above formula holds only for the eigenfields promised by the Rellich theorem; in particular, we do not get to choose the eigenfield V. By contrast, the corresponding formula in Theorem C holds for all the eigenfields V with eigenvalue $\lambda(\Omega)$.

In the proof of the above first variation formula, we replace the various modified Biot–Savart operators BS'_t by their inverse curl operators, and then pull all these operators back to the fixed domain Ω to permit application of the Rellich perturbation theorem. We will not use the above formula in this paper, and so omit its proof.

K. Proof of Theorem D

Now we suppose that the vector field V on the compact, smoothly bounded domain Ω maximizes helicity among all divergence-free vector fields of given nonzero energy, defined on and tangent to the boundary of all such domains of given volume in three-space.

We must show that

(1) $|V|$ is a nonzero constant on the boundary of each component of Ω.
(2) All the components of $\partial\Omega$ are tori.
(3) The orbits of V are geodesics on $\partial\Omega$.

To start, the fact that V maximizes helicity for given energy on Ω tells us that V must be an eigenfield of the modified Biot–Savart operator BS′ corresponding to the largest eigenvalue $\lambda(V)=\lambda(\Omega)$.

Furthermore, the fact that V on Ω maximizes helicity for given energy among all domains having the same volume as Ω tells us that $\delta\lambda(V)=0$ for all volume-preserving deformations of Ω. Otherwise there would be a volume-preserving deformation of Ω for which $\delta\lambda(V)>0$. Then by part 3 of Theorem C, we would have $\delta\lambda(\Omega)>0$, contrary to assumption.

We must also have $\delta E(V)=0$ for all volume-preserving deformations, since

$$\delta\lambda(V)=\lambda(V)(-\delta E(V)/E(V)).$$

Then from Corollary 3 to Theorem I we learn that the orbits of V are constant speed geodesics on $\partial\Omega$, while from Corollary 5 we see that $|V|$ must be constant on the boundary of each component of Ω.

It remains to see why each of these constants must be nonzero. Once this is in hand, it will follow immediately that all the components of $\partial\Omega$ are tori.

Vainshtein's lemma[19] (1992): *Suppose the vector field V defined on the compact, smoothly bounded domain Ω is divergence-free and an eigenfield of curl.*

If $V\equiv 0$ on $\partial\Omega$, then $V\equiv 0$ throughout Ω.

Proof: Following Vainshtein, we define the vector field

$$U=\tfrac{1}{2}|V|^2\mathbf{r}-(\mathbf{r}\cdot V)V,$$

where $\mathbf{r}$ is the position vector field in three-space, and will show in the following sublemma that $\nabla\cdot U=\frac{1}{2}|V|^2$ as a consequence of the hypotheses that V is divergence-free and an eigenfield of curl.

Assuming this for the moment, we then have

$$\int_\Omega \tfrac{1}{2}|V|^2\, d(\mathrm{vol})=\int_\Omega \nabla\cdot U\, d(\mathrm{vol})=\int_{\partial\Omega} U\cdot n\, d(\mathrm{area})=0,$$

since $U\equiv 0$ on $\partial\Omega$.

Thus $V\equiv 0$ throughout Ω, as claimed.

Sublemma: *Let V be any vector field, and define the vector field U by*

$$U=\tfrac{1}{2}|V|^2\mathbf{r}-(\mathbf{r}\cdot V)V.$$

Then

$$\nabla\cdot U=\tfrac{1}{2}|V|^2+(V\times(\nabla\times V))\cdot\mathbf{r}-(\mathbf{r}\cdot V)(\nabla\cdot V).$$

Hence if V is divergence-free and an eigenfield of curl, we get

$$\nabla\cdot U=\tfrac{1}{2}|V|^2.$$

Proof: The argument seems to us a bit clumsy in the notation we have been using throughout this paper, but effortless in subscript notation with respect to rectangular coordinates.

In that notation, the vector $A=(a_1,a_2,a_3)$ in rectangular coordinates is simply recorded as a_i. Thus the position vector $\mathbf{r}=(x_1,x_2,x_3)$ appears as x_i.

Summation convention over repeated indices is employed, so that

$$A \cdot B = a_i b_i .$$

The partial derivative $\partial v_i / \partial x_j$ is recorded as $v_{i,j}$, and thus

$$\nabla \cdot V = v_{i,i} .$$

In this style, the triple vector product is given by

$$(A \times B) \cdot C = \sigma_{ijk} a_i b_j c_k ,$$

where σ_{ijk} takes the value 1 if ijk is an even permutation of 123, the value -1 if it is an odd permutation of 123, and the value 0 otherwise.

Finally, the curl appears as

$$\nabla \times V = \sigma_{ijk}(v_{j,i} - v_{i,j}).$$

With this notation, the proof simply flows:

$$U = u_i = \tfrac{1}{2} v_j v_j x_i - x_j v_j v_i ,$$

$$\nabla \cdot U = u_{i,i} = v_j v_{j,i} x_i + \tfrac{1}{2} v_j v_j x_{i,i} - x_{j,i} v_j v_i - x_j v_{j,i} v_i - x_j v_j v_{i,i} .$$

Now the divergence $x_{i,i}$ of the position vector $\mathbf{r}$ is 3 and the partial derivative $x_{j,i}$ is 1 if $j = i$ and 0 if $j \neq i$, so our expression for $\nabla \cdot U$ simplifies to

$$\begin{aligned}
\nabla \cdot U &= v_j v_{j,i} x_i + \tfrac{3}{2} v_j v_j - v_j v_j - x_j v_{j,i} v_i - x_j v_j v_{i,i} \\
&= \tfrac{1}{2} v_j v_j + v_j v_{j,i} x_i - v_i v_{j,i} x_j - x_j v_j v_{i,i} \\
&= \tfrac{1}{2} v_j v_j + v_j (v_{j,i} - v_{i,j}) x_i - x_j v_j v_{i,i} ,
\end{aligned}$$

where the last line is obtained by interchanging the subscripts i and j in the third term of the line above it.

This is exactly the formula we want: the first term on the last line above is $\frac{1}{2}|V|^2$, the second can be recognized as the triple product $(V \times (\nabla \times V)) \cdot \mathbf{r}$ by using the subscript formulas for curl and triple product, and the third term is $(\mathbf{r} \cdot V)(\nabla \cdot V)$.

This completes the proof of the sublemma and, with it, that of Vainshtein's lemma.

L. Conclusion of the proof of Theorem D

We have already seen that $|V|$ must be constant on the boundary of each component of Ω, and are left with the task of showing that each of these constants must be nonzero.

At the beginning of the proof, we noted that if V satisfies the hypotheses of Theorem D, then it must be an eigenfield of the modified Biot–Savart operator BS′. Hence, as we saw earlier, it must also be an eigenfield of curl. Therefore V, since it is divergence-free, satisfies the hypotheses of Vainshtein's lemma.

Suppose that the constant value of $|V|$ on the boundary of the component Ω_i of Ω is zero. Apply Vainshtein's lemma to that component to conclude that V must be identically zero throughout Ω_i.

Since V has nonzero energy by hypothesis, there must be other components of Ω where V does not vanish. Write $\Omega = \Omega_1 \cup \Omega_2$, where Ω_1 is the union of the components of Ω where V does not vanish, and Ω_2 is the union of the components where V does vanish. We intend to replace Ω by a scaled up version of Ω_1.

To do this, delete Ω_2 and multiply Ω_1 by the factor $k > 1$ so that

$$\mathrm{vol}\, k\Omega_1 = k^3 \,\mathrm{vol}\, \Omega_1 = \mathrm{vol}\, \Omega;$$

carry the vector field V on Ω_1 along with the expansion to give the vector field kV on $k\Omega_1$. Then a glance at the formulas for helicity and energy shows that

$$H(kV)=k^6H(V) \quad \text{while} \quad E(kV)=k^5E(V).$$

Hence the ratio of helicity to energy has increased by the factor $k>1$, contrary to the hypothesis that the original vector field V on Ω maximized helicity for given energy and given volume of domain.

It follows that V cannot vanish on any of the components of Ω, and hence that on the boundary of each of these components, $|V|$ must be a nonzero constant.

Then each boundary component of Ω, since it supports a nowhere-vanishing vector field, must have Euler characteristic zero, and hence be a torus.

We saw earlier that the orbits of V are geodesics on $\partial\Omega$, and so we are now finished with the proof of Theorem D.

M. Optimal domains

The goal of the isoperimetric problem in the setting of this paper is to maximize helicity among all divergence-free vector fields of given energy, defined on and tangent to the boundary of all domains of given volume in three-space.

Theorem E provides an upper bound for these helicities.

Theorem D tells us some features of an optimal (that is, helicity-maximizing) domain, and of the helicity-maximizing vector field on it.

But how do we find such a domain?

Suppose we begin with the vector field V which maximizes helicity for given nonzero energy on a round ball Ω, the Woltjer spheromak field described earlier and pictured in Fig. 2.

We seek a volume-preserving deformation of Ω which increases $\lambda(\Omega)$, guided by the inequality of Theorem C:

$$\delta\lambda(\Omega)\geqslant\lambda(\Omega)\frac{\int_{\partial\Omega}|V|^2(W\cdot n)\,d(\text{area})}{\int_{\Omega}|V|^2\,d(\text{vol})}.$$

We maximize the right-hand side by choosing

$$W\cdot n=|V|^2-\text{average value of }|V|^2\text{on }\partial\Omega.$$

Then we imagine a volume-preserving deformation of Ω whose initial velocity field W has this preassigned normal component along the boundary. The deformation begins by dimpling Ω inwards near the poles and bulging it outwards near the equator, making the ball look somewhat like an apple. We repeat this calculation at each stage of the deformation, trying to follow a path of steepest ascent for the largest eigenvalue of the modified Biot–Savart operator.

We believe that this procedure will continue to dimple the apple inwards at the poles and bulge it outwards at the equator, until it reaches roughly the shape pictured in Fig. 5, which then maximizes the largest eigenvalue $\lambda(\Omega)$ of the modified Biot–Savart operator among all domains of given volume. We think of this domain either as an extreme apple, in which the north and south poles have been pressed together, or as an extreme solid torus, in which the hole has been shrunk to a point. We also show in Fig. 5 the expected appearance of the helicity-maximizing vector field.

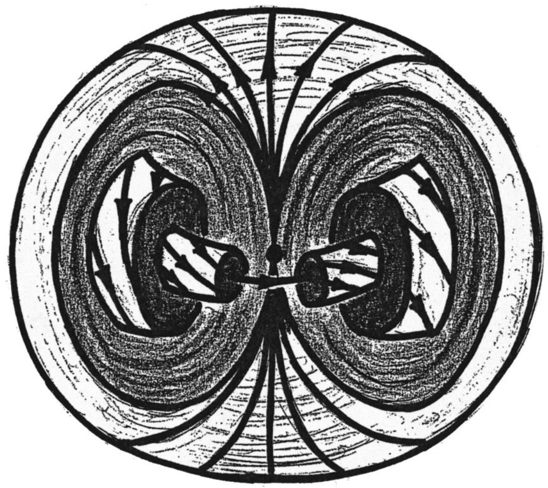

FIG. 5. The expected optimal domain and field.

Comparison of this picture with those of the helicity maximizers on the flat solid torus and on the round ball, given in Figs. 1 and 2, shows that we expect the common underlying pattern to persist even as the domain becomes singular.

A computational search for this singular optimal domain and the helicity-maximizing vector field on it is at present under way, guided by a discrete version of the evolution described above.

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