TOPOLOGICAL STRUCTURE OF STABLE PLASMA FLOWS

Jason Cantarella

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Dedication

To Heather Johnston, the bright star of my heart.

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ABSTRACT

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Jason Cantarella

Herman Gluck

A plasma flow, confined to a vessel in 3-space, quickly relaxes to a configuration with the least energy compatible with its helicity, a measure of the wrapping and coiling of the orbits of the field around each other. We study these energy-minimizing flows on axisymmetric solid tori representing possible "tokamak" plasma confinement devices, proving that they are first eigenfields of curl with certain boundary conditions. Exploiting the rotational symmetry of the problem, using Rayleigh quotient arguments, and deriving a new formula for the "cross-helicity" of flows on disjoint domains in 3-space, we show that when these fields are axisymmetric, their fundamental topological structure depends very little on the geometry of the vessel. Each of these fields has a positive component in the direction of rotation inside the vessel, and no component in that direction on the boundary. These flows have no stationary points, on the interior or on the boundary. Each is tangent to a family of integral tori, and every torus is smoothly immersed, while most are smoothly embedded. Singular integral surfaces are rare (we conjecture that they do not occur on domains of convex cross-section), and when they do appear their topology is strictly controlled. On the nonsingular integral tori, the orbits of these fields are roughly helical, and the helices are always right-handed. Corresponding results should apply to axisymmetric balls, representing possible "spheromak" devices, and we hope to extend these results to that case in a following paper.

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CHAPTER 1

Introduction

Physicists have been interested in the structure of plasma flows confined to domains in 3-space for many years. The domains studied in the laboratory are always rotationally symmetric, with the topology of solid tori (the "tokomak") or solid balls (the "spheromak"). Theory and experiment agree that after a short time, these plasma flows minimize their energy, subject to the constraint of fixed helicity, a measure of the wrapping and coiling of the orbits of the field around one another. These energy-minimizing flows seem to have a characteristic structure.

For example, consider the energy-minimizer on the "round" solid torus:

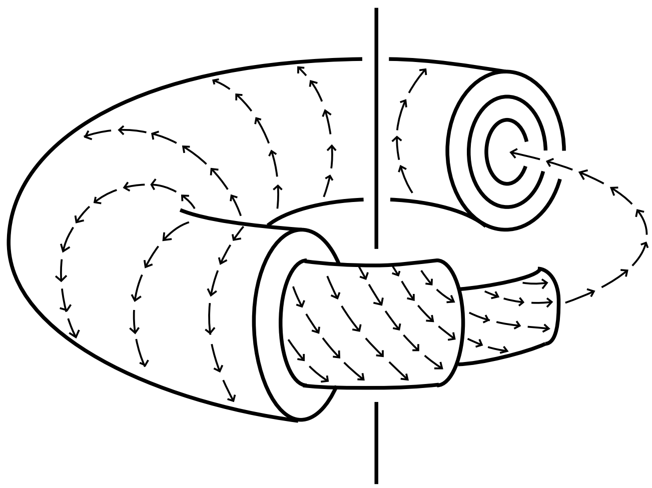


Figure 1: The energy-minimizer on a "round" solid torus.

or on the unit ball:

Each of these vector fields is tangent to a family of integral tori, with a single core orbit. The field on the solid torus never vanishes, while the field on the ball vanishes only on the boundary. On each integral torus, the orbits of each field form right-handed helices. These helices vary in pitch as the integral tori expand from the core orbit to the boundary, and the vector fields have no component in the direction of revolution on the boundary itself. On each integral torus, either all the orbits are closed, or none are.

The goal of our work will be to examine the structure of rotationally symmetric, energy minimizing vector fields of fixed helicity on domains of revolution in 3-space, and to prove that the features above are characteristic of such fields.

Let Ω be a compact domain of revolution in 3-space with smooth boundary. For now, we restrict our attention to the case where Ω represents a tokamak. That is, we assume

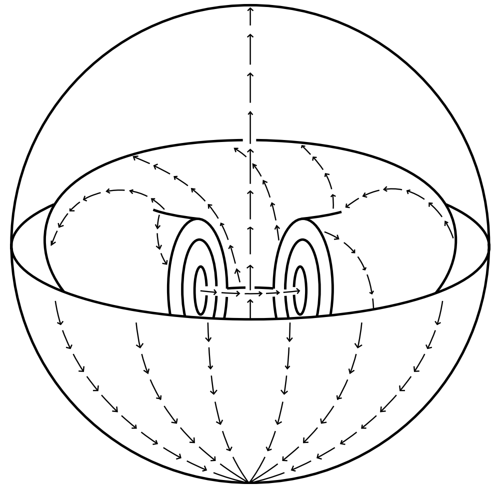


Figure 2: The energy-minimizer on a round ball.

that Ω has the topology of a solid torus. We also restrict our attention to the space of axisymmetric fluid knots, or divergence-free vector fields tangent to the boundary of Ω:

$$\mathrm{SK}(\Omega) = \left\{ \begin{array}{c} V \text{ is an axisymmetric} \\ \text{smooth vector field on } \Omega \end{array} \middle| \nabla \cdot V = 0, V \cdot n = 0 \right\}.$$

We regard $\mathrm{SK}(\Omega)$ as an infinite-dimensional vector space, with the L^2 inner product $\langle V, W \rangle = \int_\Omega V \cdot W \,\mathrm{dVol}$. Under these conditions, we will prove

THEOREM 4. *Let V be a vector field in the L^2 closure of $\mathrm{SK}(\Omega)$ which has minimum energy among all such vector fields of fixed positive helicity. Then V is an eigenfield of the modified Biot-Savart operator BS' with the largest eigenvalue among all such vector fields.*

The Biot-Savart operator BS computes the magnetic field generated by a current flow. While the current field is divergence-free and tangent to the boundary of Ω, the magnetic field is only divergence-free. The modified Biot-Savart operator subtracts a gradient from the magnetic field to arrive at a field inside $\mathrm{SK}(\Omega)$. On the space $\mathrm{SK}(\Omega)$, the curl operator is a left inverse for BS'. We will show that the image of BS' is the space of axisymmetric "Amperian" knots:

$$\mathrm{SAK}(\Omega) = \left\{ V \in \mathrm{SK}(\Omega) \middle| \begin{array}{c} \int_C V \cdot \mathrm{ds} = 0 \text{ for every curve } C \text{ in } \partial\Omega \\ \text{which bounds outside } \Omega \end{array} \right\}.$$

We can then trade the integral operator BS' for the differential operator curl:

COROLLARY 1. *Let V be the eigenfield of curl in $\mathrm{SAK}(\Omega)$ with least positive eigenvalue. Then V is the vector field in $\mathrm{SK}(\Omega)$ with least energy among all such vector fields of fixed positive helicity.*

We will use rotational symmetry to reduce the eigenvalue problem for curl on $\mathrm{SAK}(\Omega)$ to the eigenvalue problem for a certain elliptic operator L on a cross-section of Ω, with Dirichlet boundary conditions.

A number of facts about the first eigenfield of curl will follow from this reduction. In particular,

THEOREM 5. *If* $V = u(r,z)\hat{r} + v(r,z)\hat{\phi} + w(r,z)\hat{z}$ *is the first eigenfield of curl in* $\mathrm{SAK}(\Omega)$*, for an axisymmetric solid torus* Ω*, then* $v(r,z)$ *is strictly positive inside* Ω*.*

THEOREM 7. *Suppose* Ω *is a rotationally symmetric solid torus with smooth boundary. The first eigenfield of curl on* $\mathrm{SAK}(\Omega)$ *never vanishes on* Ω *or on its smooth boundary.*

We will also be able to show that every eigenfield of curl in $\mathrm{SAK}(\Omega)$ organizes itself into a family of integral surfaces:

THEOREM 9. *Using the cylindrical coordinates* r*,* ϕ*, and* z*, every axisymmetric eigenfield*

$$V = u(r,z)\hat{r} + v(r,z)\hat{\phi} + w(r,z)\hat{z}$$

of curl is tangent to the level surfaces of the function rv*.*

This is just conservation of angular momentum. We will then determine the structure of these integral surfaces.

THEOREM 10. *Let* V *be the first eigenfield of curl in* $\mathrm{SAK}(\Omega)$*. Every nonsingular integral surface of* V *is a rotationally symmetric, smoothly embedded torus. Except for* $\partial\Omega$*, every singular integral surface of* V *is a rotationally symmetric, smoothly immersed torus.*

Suppose we have an immersed circle C in the plane, enclosing a collection of bounded open domains, as below left. We can construct the graph $G(C)$ of this circle by assigning a vertex to each open domain, and joining these vertices with edges when the corresponding regions share boundary points, as below right.

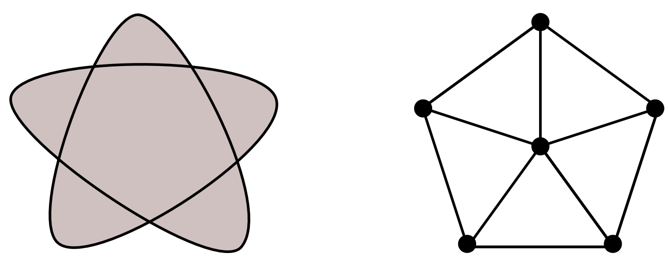


Figure 3: An immersed circle and its graph.

THEOREM 11. *Let* V *be the first eigenfield of curl in* $\mathrm{SAK}(\Omega)$*, and let* C *be the cross section of a singular integral surface of* V*. Then the graph* $G(C)$ *is a tree.*

The content of this theorem is summarized by the pictures below.

On the axisymmetric solid torus of circular cross-section, there were no singular integral surfaces in the first eigenfield of curl in SAK. This cannot be true on all axisymmetric solid tori– the first eigenfield of curl in SAK on the solid torus with dumbell-shaped cross-section has a singular integral surface whose cross-section is a figure eight:

It seems that the true cause of singular integral surfaces is the failure of the cross-section to be convex. Numerical evidence strongly supports the conjecture that the first eigenfield of curl on a domain with convex cross-section has no singular integral surfaces.

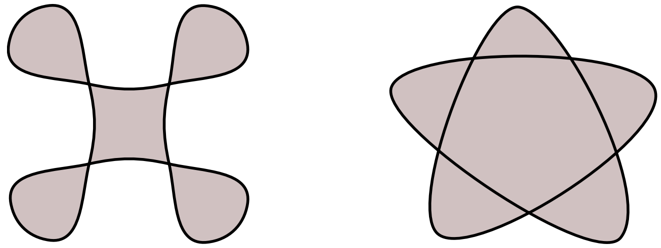


Figure 4: Permitted and forbidden cross-sections.

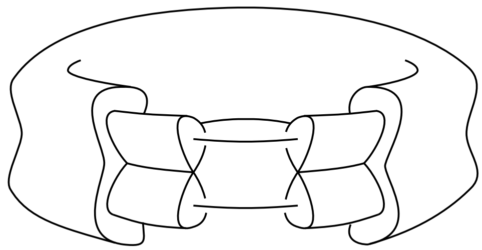


Figure 5: A solid torus with singular integral surfaces.

I cannot yet prove this conjecture, but I can prove an improved structure theorem for the singular integral surfaces of V if the failure of the cross-section to be convex is controlled.

THEOREM 12. *Let $C(\Omega)$ be a cross-section in an r-z plane of the axisymmetric solid torus Ω. Suppose $C(\Omega)$ has a flip symmetry over the r axis, and that the normal vector to $\partial C(\Omega)$ points in the $\hat{r}$ direction only where $\partial C(\Omega)$ crosses the r-axis, as below.*

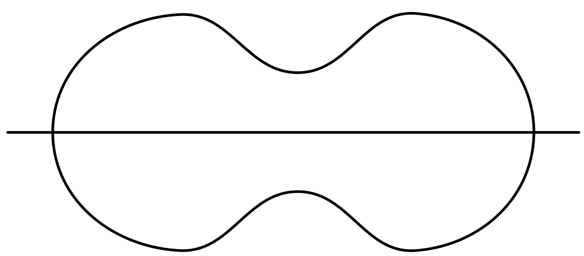


Figure 6: Cross-section allowed by these conditions.

Let V be the first eigenfield of curl in $\mathrm{SAK}(\Omega)$. Then the cross-section of every singular integral surface is one of the "multiple eights" below.

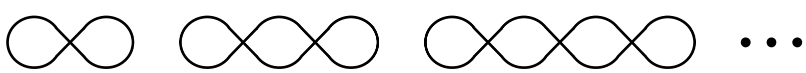


Figure 7: Possible cross-sections for singular leaves.

Having found out as much as we can about the structure of the integral surfaces of V, we will focus on the behavior of V on each individual surface. Using a "Rayleigh quotient" argument, we will be able to prove:

THEOREM 14. *Let V be the first eigenfield of curl with* SAK *boundary conditions. Then on each nonsingular integral torus, the orbits of V are roughly helical, and these helices are always right-handed.*

Furthermore, on each integral torus, either all the orbits of V are closed, or none are.

To prove this Theorem, we will need a formula for the helicity of a divergence-free vector field confined to a pair of disjoint domains Ω and Ω' in 3-space, and tangent to their boundaries. No such formula exists in the literature without strict conditions on the geometry and topology of Ω and Ω'.

We will derive a formula for this helicity, in terms of the linking numbers of curves C_i and C_i' which span $H_1(\Omega, \mathbf{R})$ and $H_1(\Omega', \mathbf{R})$, and the flux of V across the Poincaré dual surfaces inside Ω and Ω'. Our formula will hold for domains of arbitrary geometry and topology. The proof of this formula is found in the last Chapter.

CHAPTER 2

Definitions, Background, and Previous Results

1. Introduction

In this chapter, we'll review some of the fundamental concepts involved in the study of energy-minimizing vector fields of fixed helicity, and state our previous results on these problems. This story summarizes a series of three papers: *Upper Bounds for the Writhing of Knots and the Helicity of Vector Fields*, *The Biot-Savart Operator for Application To Knot Theory, Fluid Dynamics and Plasma Physics*, and *Hodge Decomposition of Vector Fields on Bounded Domains in 3-Space*, written with Dennis DeTurck and Herman Gluck [**4**], [**5**], [**7**]. All of the proofs of the theorems mentioned in this Chapter may be found in these three papers. Much of this chapter comes from *The Influence of Geometry and Topology on Helicity*, written with Dennis DeTurck, Herman Gluck, and Misha Teytel [**8**].

2. Helicity

The *helicity* $\mathrm{H}(V)$ of a smooth (meaning C^∞) vector field V on the domain Ω in 3-space, defined by the formula

$$\mathrm{H}(V) = \frac{1}{4\pi} \int_{\Omega\times\Omega} V(x) \times V(y) \cdot \frac{x-y}{|x-y|^3}\, d\,\mathrm{vol}_x \; d\,\mathrm{vol}_y,$$

is the standard measure of the extent to which the field lines wrap and coil around one another. It was introduced by Woltjer in [**18**] and it was named by Moffatt in [**13**].

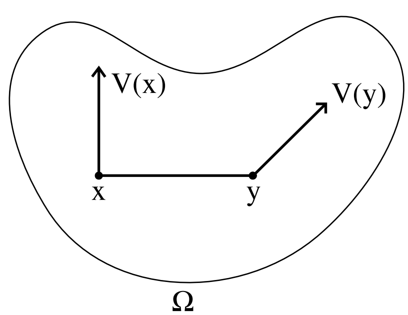


Figure 8: The Helicity integrand.

3. Relation between Helicity and Linking Number

Clearly, the definition of helicity is a variation of the integral formula of Gauss [**9**] for the linking number of two disjoint closed space curves, parametrized by arc-length:

$$\mathrm{Lk}(C_1, C_2) = \frac{1}{4\pi}\int_{C_1\times C_2} C_1'(s)\times C_2'(t)\cdot\frac{C_1(s)-C_2(t)}{|C_1(s)-C_2(t)|^3}\,ds\,dt.$$

A formula which makes this observation rigorous appears in [**16**]. Let V be a vector field defined in two tubes Ω_1, Ω_2 about a pair of space curves C_1 and C_2, orthogonal to the cross-sectional disks, with length depending only on distance from the core curve. Such a vector field is always divergence-free. Further, assume that all the orbits of V are closed, and that no pair of orbits in the same tube are linked. Then

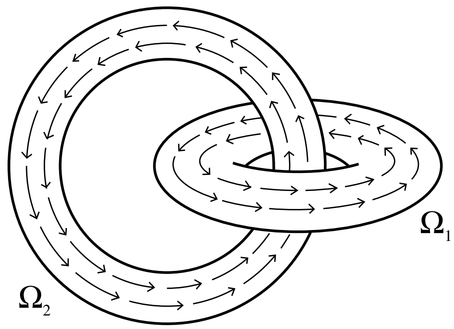


Figure 9: Linked tubes, with spanning surfaces.

$$\mathrm{H}(V) = 2\,\mathrm{Flux}(V,\Omega_1)\ \mathrm{Flux}(V,\Omega_2)\ \mathrm{Lk}(C_1, C_2).$$

Here, $\mathrm{Flux}(V,\Omega_1)$ denotes the flux of V through a cross-sectional disk in the tube Ω_1.

4. How the Geometry of the Domain Influences Helicity

THEOREM 1. *Let V be a smooth vector field defined on the compact domain Ω with smooth boundary. Then the helicity $\mathrm{H}(V)$ of V is bounded by*

$$|\,\mathrm{H}(V)| \le R(\Omega)\ \mathrm{E}(V),$$

where $R(\Omega)$ is the radius of a ball with the same volume as Ω and $\mathrm{E}(V) = \int_\Omega V\cdot V\,d\,\mathrm{vol}$ is the energy of V.

This upper bound is not sharp, but it is the right order of magnitude. For example, the vector field of unit energy with maximum helicity on the unit ball has has helicity greater than one-fifth of this upper bound. Sharp upper bounds will be obtained by the spectral methods discussed below.

5. Magnetic Fields and Helicity

Start with a vector field V on the domain Ω, regard it as a current distribution, and use the Biot-Savart Law to compute its magnetic field:

$$\mathrm{BS}(V)(y) = \frac{1}{4\pi}\int_\Omega V(x)\times\frac{y-x}{|y-x|^3}\,d\,\mathrm{vol}_x\,.$$

The helicity of V can then be expressed as the integrated dot product of V with its magnetic

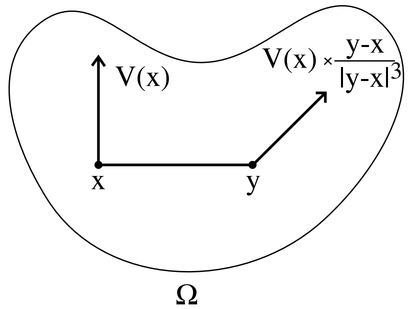


Figure 10: The Biot-Savart Law.

field $\mathrm{BS}(V)$:

$$\begin{aligned}\mathrm{H}(V) &= \frac{1}{4\pi} \int\limits_{\Omega\times\Omega} V(x) \times V(y) \cdot \frac{x-y}{|x-y|^3} \, d\,\mathrm{vol}_x \; d\,\mathrm{vol}_y \\ &= \int_\Omega V(y) \cdot \left[\frac{1}{4\pi} \int_\Omega V(x) \times \frac{y-x}{|y-x|^3} \, d\,\mathrm{vol}_x \right] d\,\mathrm{vol}_y \\ &= \int_\Omega V \cdot \mathrm{BS}(V) \, d\,\mathrm{vol}\,.\end{aligned}$$

6. A general point of view

Let Ω be a compact domain in 3-space with smooth boundary. Let $\mathrm{VF}(\Omega)$ denote the set of all smooth vector fields V on Ω. Then $\mathrm{VF}(\Omega)$ is itself an infinite-dimensional vector space.

Define an inner product on $\mathrm{VF}(\Omega)$ by the formula

$$\langle V, W \rangle = \int_\Omega V \cdot W \, d\,\mathrm{vol}\,.$$

Although the magnetic field $\mathrm{BS}(V)$ is well-defined throughout all of 3-space, we will restrict it to Ω; thus the Biot-Savart Law provides an operator

$$\mathrm{BS} : \mathrm{VF}(\Omega) \to \mathrm{VF}(\Omega).$$

Using the above inner product notation, our formula for the helicity of V can be written

$$\mathrm{H}(V) = \langle V, \mathrm{BS}(V) \rangle.$$

7. The Modified Biot-Savart Operator

Let $\mathrm{K}(\Omega)$ be the subspace of $\mathrm{VF}(\Omega)$ of all smooth divergence-free vector fields defined on Ω and tangent to its boundary.

Start with a vector field V in $\mathrm{K}(\Omega)$ and compute its magnetic field, $\mathrm{BS}(V)$. Restrict $\mathrm{BS}(V)$ to Ω and subtract a gradient vector field so as to keep it divergence-free while making it tangent to $\partial\Omega$. Call the resulting vector field $\mathrm{BS}'(V)$. The Hodge Decomposition

Theorem (to follow) tells us that the gradient vector fields on Ω form the orthogonal complement of $\mathrm{K}(\Omega)$; hence $\mathrm{BS}'(V)$ can be viewed as the orthogonal projection of $\mathrm{BS}(V)$ back into $\mathrm{K}(\Omega)$.

The *modified Biot-Savart operator*

$$\mathrm{BS}' : \mathrm{K}(\Omega) \to \mathrm{K}(\Omega),$$

will play a leading role in our story.

The helicity of a vector field V in $K(\Omega)$ is given by

$$\mathrm{H}(V) = \langle V, \mathrm{BS}'(V) \rangle \,,$$

since $\mathrm{BS}(V)$ and $\mathrm{BS}'(V)$ differ by a gradient vector field, which as we just noted is orthogonal in the inner product structure of $\mathrm{VF}(\Omega)$ to any vector field V in $\mathrm{K}(\Omega)$.

8. Spectral Methods

From now on, we focus on vector fields which are divergence-free and tangent to the boundary of their domain, that is, on the subspace $\mathrm{K}(\Omega)$ of $\mathrm{VF}(\Omega)$, and on the modified Biot-Savart operator $\mathrm{BS}' : \mathrm{K}(\Omega) \to \mathrm{K}(\Omega)$. A standard functional analysis argument yields

THEOREM 2. *The modified Biot-Savart operator* BS' *is a bounded operator, and hence extends to a bounded operator on the* L^2 *completion of its domain; there it is both compact and self-adjoint.*

The *Spectral Theorem* then promises that BS' behaves like a real self-adjoint matrix: the L^2 completion of its domain admits an orthonormal basis of eigenfields, in terms of which the operator is "diagonalizable". The eigenfields corresponding to the eigenvalues $\lambda(\Omega)$ of maximum absolute value have minimum energy for fixed helicity, and we obtain the sharp upper bound

$$|\,\mathrm{H}(V)| \leq |\lambda(\Omega)|\; \mathrm{E}(V),$$

for all V in $\mathrm{K}(\Omega)$.

This approach to the study of helicity was initiated by Arnold in [**1**] for the setting of closed orientable 3-manifolds. For a corresponding approach via the curl operator on domains in Euclidean space, see [**19**] and [**11**].

9. The Connection with the Curl Operator

If the vector field V is divergence-free and tangent to the boundary of its domain Ω, that is, if V is in $\mathrm{K}(\Omega)$, then

$$\nabla \times \mathrm{BS}(V) = V.$$

Since $\mathrm{BS}(V)$ and $\mathrm{BS}'(V)$ differ by a gradient vector field, we also have

$$\nabla \times \mathrm{BS}'(V) = V.$$

If V is an eigenfield of BS',

$$\mathrm{BS}'(V) = \lambda\, V,$$

then

$$\nabla \times V = \frac{1}{\lambda} V.$$

Thus the eigenvalue problem for BS' can be converted to an eigenvalue problem for curl on the image of BS', which means to a system of partial differential equations. Even though we extended BS' to the L^2 completion of $\mathrm{K}(\Omega)$ in order to apply the spectral theorem, the eigenfields are smooth vector fields in $K(\Omega)$; this follows, thanks to elliptic regularity, because on divergence-free vector fields, the square of the curl is the negative of the Laplacian. Hence these vector fields can be (and are) discovered by solving the above system of PDEs.

10. Explicit Computation of Energy Minimizers

We solve $\nabla \times V = (1/\lambda)V$ on the flat solid torus $D^2(a) \times S^1$, where $D^2(a)$ is a disk of radius a and S^1 is a circle of any length; see [**4**]. Although this is not a subdomain of 3-space, the solution here is so clear-cut and instructive as to be irresistible.

The eigenvalues of BS' of largest absolute value are

$$\lambda(D^2(a) \times S^1) = \pm\frac{a}{2.405...},$$

where the denominator is the first positive zero of the Bessel function J_0, and the corresponding eigenfields, discovered in [**12**], are

$$V_\lambda = J_1(r/\lambda)\,\hat{\varphi} + J_0(r/\lambda)\,\hat{z},$$

expressed in terms of cylindrical coordinates (r, φ, z) and the Bessel functions J_0 and J_1.

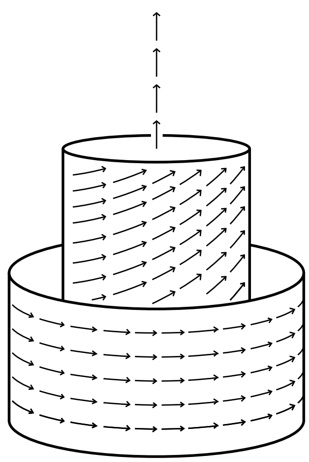


Figure 11: The Lundquist field.

It follows that for any V in $K(D^2(a) \times S^1)$,

$$|\,\mathrm{H}(V)| \leq \frac{a}{2.405..}\;\mathrm{E}(V),$$

with equality for the eigenfield V_λ.

We solve $\nabla \times V = (1/\lambda)V$ on the round ball $B^3(a)$ of radius a in terms of spherical Bessel functions in [**6**] .

The eigenvalues of BS' of largest absolute value are

$$\lambda(B^3(a)) = \pm\frac{a}{4.4934...},$$

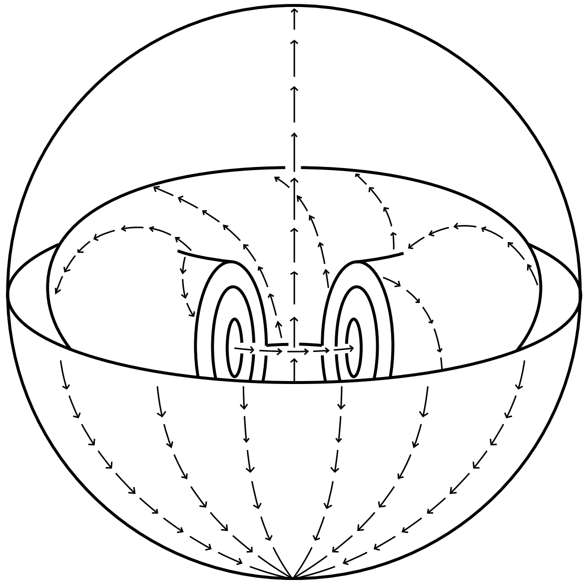


Figure 12: The energy-minimizer on a round ball.

where the denominator is the first positive zero of $(\sin x)/x - \cos x$. The corresponding eigenfields V_λ, representing plasma flows in the round spheromak, are also Woltjer's models for the magnetic field in the Crab Nebula. In spherical coordinates (r, θ, φ) on a ball of radius $a = 1$,

$$V_\lambda(r, \theta, \varphi) = u(r, \theta)\,\hat{r} + v(r, \theta)\,\hat{\theta} + w(r, \theta)\,\hat{\varphi},$$

where

$$u(r, \theta) = \frac{2\lambda}{r^2}\left(\frac{\sin(r/\lambda)}{r/\lambda} - \cos(r/\lambda)\right)\cos\theta,$$

$$v(r, \theta) = -\frac{1}{r}\left(\frac{\cos(r/\lambda)}{r/\lambda} - \frac{\sin(r/\lambda)}{(r/\lambda)^2} + \sin(r/\lambda)\right)\sin\theta,$$

$$w(r, \theta) = \frac{1}{r}\left(\frac{\sin(r/\lambda)}{r/\lambda} - \cos(r/\lambda)\right)\sin\theta.$$

The values $\lambda = \pm 1/4.4934...$ make both $u(r, \theta)$ and $w(r, \theta)$ vanish when r=1, that is, at the boundary of the ball. As a consequence, the vector field V_λ is tangent to the boundary of the ball, and directed there along the meridians of longitude.

It follows that for any V in $K(B^3(a))$,

$$|\,\mathrm{H}(V)| \leq \frac{a}{4.4934...}\,\mathrm{E}(V),$$

with equality for the eigenfield V_λ.

Compare this with the rough upper bound from Theorem 1:

$$|\,\mathrm{H}(V)| \leq a\,\mathrm{E}(V).$$

11. Higher Eigenfields of BS'

In order to train our intuition, we now examine some of the higher eigenfields of curl on these model domains. On the flat solid torus of unit radius, one family of eigenfields is given in cylindrical coordinates by the equations:

$$V_\lambda = J_1(r/\lambda)\,\hat{\varphi} + J_0(r/\lambda)\,\hat{z},$$

expressed in terms of cylindrical coordinates (r, φ, z) and the Bessel functions J_0 and J_1, where $1/\lambda$ is a positive zero of J_0. For instance, if $1/\lambda$ is the second positive zero of J_0, the eigenfield appears as below:

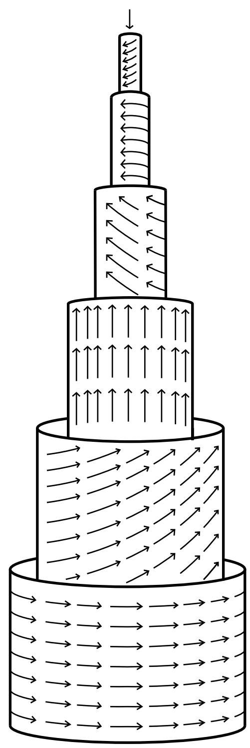


Figure 13: The second eigenfield of curl on a flat solid torus.

Notice the interior leaf where V_λ is vertical, and the region where V_λ points down. We will later prove that these features are characteristic of higher eigenfields of BS', and can never occur in a first axisymmetric eigenfield.

Another useful example is the second axisymmetric eigenfield of BS' on the round ball, given in spherical coordinates by

$$V_\lambda(r, \theta, \varphi) = u(r, \theta)\,\hat{r} + v(r, \theta)\,\hat{\theta} + w(r, \theta)\,\hat{\varphi},$$

where

$$w(r, \theta) = r^{-1/2} J_{5/2}(\lambda r) P_2^1(\cos\theta),$$

and

$$u(r, \theta) = \frac{1}{\lambda \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta w) \text{ and } v(r, \theta) = -\frac{1}{\lambda r} \frac{\partial}{\partial r}(rw).$$

where $J_{5/2}$ is the Bessel function, P_2^1 is the associated Legendre polynomial of the first kind, defined by $P_2^1(x) = \sqrt{1-x^2}P_2'(x)$, and $\lambda \sim 5.763\ldots$ is the first positive zero of the Bessel function $J_{5/2}$.

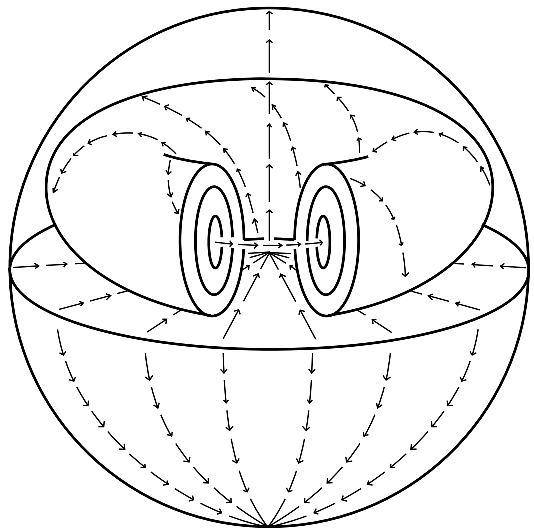


Figure 14: The second eigenfield of curl on a round ball.

Notice the horizontal floor dividing the ball into upper and lower hemispheres, each of which contains a family of integral tori. Notice also that V_λ vanishes on the equator of the sphere. We will later see that these features are also characteristic of higher eigenfields of BS', and can never occur in a first eigenfield.

12. The Hodge Decomposition Theorem for Vector Fields

We now turn to the description of one last tool for working with vector fields on domains in 3-space: the Hodge Decomposition Theorem. In a multivariable calculus course, we are taught that the topology of the underlying domain affects the calculus of vector fields defined on it. For example, we learn that to test whether a vector field is the gradient of a function, we must take its curl and see if it is zero. If the curl is not zero, then the vector field is certainly not a gradient. If the curl is zero and the domain is simply connected, we learn that the vector field is a gradient. But if the curl is zero and the domain is not simply connected, then we learn that the vector field may or may not be a gradient, and that further tests are required.

The Hodge Decomposition Theorem for vector fields on domains in 3-space provides a more sophisticated level of control over this same subject.

The following two questions help to set the mood.

Question 1. *Is there a nonzero vector field V on the domain which is divergence-free, curl-free and tangent to the boundary?*

Question 2. *Is there a nonzero gradient vector field V on the domain which is divergence-free and orthogonal to the boundary?*

The answers to these questions can be found in Table 1.

Domain	Answers to Question	
	1	2
Ball	No	No
Solid torus	Yes	No
Spherical shell	No	Yes
Toroidal shell	Yes	Yes

Table 1: Answers to motivating questions.

13. The Hodge Decomposition Theorem

Let Ω be a compact domain with smooth boundary in 3-space.

The following is arguably the single most useful expression of the interplay between the topology of the domain Ω, the traditional calculus of vector fields defined on this domain, and the inner product structure on $\mathrm{VF}(\Omega)$ defined by the formula $\langle V, W \rangle = \int_\Omega V \cdot W \, d\,\mathrm{vol}$. [**3**] and [**17**] are good references; a detailed treatment and proof of this theorem in the form given below appears in our paper [**7**].

THEOREM 3. *We have a direct sum decomposition of* $\mathrm{VF}(\Omega)$ *into five mutually orthogonal subspaces,*

$$\mathrm{VF}(\Omega) = \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG},$$

with

$$\begin{aligned}
\mathrm{Ker\,Curl} &= \phantom{\mathrm{FK} \oplus{}} \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG}\\
\mathrm{Image\,Grad} &= \phantom{\mathrm{FK} \oplus \mathrm{HK} \oplus{}} \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG}\\
\mathrm{Image\,Curl} &= \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG}\\
\mathrm{Ker\,Div} &= \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG}
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{FK} &= \{\nabla\cdot V = 0,\ V\cdot n = 0,\ \text{all interior fluxes } = 0\},\\
\mathrm{HK} &= \{\nabla\cdot V = 0, \nabla\times V = 0, V\cdot n = 0\},\\
\mathrm{CG} &= \{V = \nabla\varphi, \nabla\cdot V = 0, \text{all boundary fluxes} = 0\},\\
\mathrm{HG} &= \{V = \nabla\varphi, \nabla\cdot V = 0, \varphi \text{ loc. constant on } \partial\Omega\},\\
\mathrm{GG} &= \{V = \nabla\varphi,\ \varphi|_{\partial\Omega} = 0\},
\end{aligned}$$

and furthermore,

$$\begin{aligned}
\mathrm{HK} &\cong H_1(\Omega;\mathbf{R}) \cong H_2(\Omega,\partial\Omega;\mathbf{R})\\
&\cong \mathbf{R}^{\text{genus of } \partial\Omega}.\\
\mathrm{HG} &\cong H_2(\Omega;\mathbf{R}) \cong H_1(\Omega,\partial\Omega;\mathbf{R})\\
&\cong \mathbf{R}^{(\#\text{ components of } \partial\Omega)-(\#\text{ components of } \Omega)}.
\end{aligned}$$

We need to explain the meanings of the conditions which appear in the statement of this theorem.

The outward pointing unit vector field orthogonal to $\partial\Omega$ is denoted by n, so the condition $V\cdot n = 0$ indicates that V is tangent to $\partial\Omega$.

Let Σ stand generically for any smooth surface in Ω with $\partial\Sigma \subset \partial\Omega$. Orient Σ by picking one of its two unit normal vector fields n. Then, for any vector field V on Ω, the *flux* of V through Σ is the value of the integral $\Phi = \int_\Sigma V\cdot n \; d\,\mathrm{Area}$.

If V is divergence-free and tangent to $\partial\Omega$, then the value of this flux depends only on the homology class of Σ in the relative homology group $H_2(\Omega,\partial\Omega;\mathbf{R})$. For example, if Ω is an n-holed solid torus, then there are disjoint oriented cross-sectional disks $\Sigma_1,\dots,\Sigma_n$, positioned so that cutting Ω along these disks will produce a simply-connected region. The fluxes $\Phi_1,\dots,\Phi_n$ of V through these disks determine the flux of V through any other cross-sectional surface.

If the flux of V through every smooth surface Σ in Ω with $\partial\Sigma\subset\partial\Omega$ vanishes, we say *all interior fluxes* $= 0$. Thus the subspace of vector fields V in $\mathrm{VF}(\Omega)$ which have

$$\nabla\cdot V = 0,\ V\cdot n = 0,\ \text{and all interior fluxes} = 0,$$

is called the subspace FK of *fluxless knots.*

The subspace HK of vector fields V in $\mathrm{VF}(\Omega)$ with

$$\nabla\cdot V = 0,\ \nabla\times V = 0,\ V\cdot n = 0,$$

called *harmonic knots*, is isomorphic to the homology group $H_1(\Omega;\mathbf{R})$ and also by Poincaré duality to the relative homology group $H_2(\Omega,\partial\Omega;\mathbf{R})$. It is thus a finite-dimensional vector space, with dimension equal to the (total) genus of $\partial\Omega$.

The orthogonal direct sum of these two subspaces,

$$\mathrm{K}(\Omega) = \mathrm{FK} \oplus \mathrm{HK},$$

is the subspace of $\mathrm{VF}(\Omega)$ mentioned earlier, consisting of all divergence-free vector fields defined on Ω and tangent to its boundary.

If V is a vector field defined on Ω, we will say that *all boundary fluxes of V are zero* if the flux of V through each component of $\partial\Omega$ is zero. The subspace of V in $\mathrm{VF}(\Omega)$ with

$$V = \nabla\varphi,\ \nabla\cdot V = 0, \text{ all boundary fluxes = 0}$$

is called the subspace CG of *curly gradients*, because these are the only gradients which lie in the image of curl.

The subspace HG of *harmonic gradients* consists of all V in $\mathrm{VF}(\Omega)$ such that

$$V = \nabla\varphi,\ \nabla\cdot V = 0,\ \varphi \text{ locally constant on } \partial\Omega,$$

meaning that φ is constant on each component of $\partial\Omega$. This subspace is isomorphic to the absolute homology group $H_2(\Omega;\mathbf{R})$ and also, via Poincaré duality, to the relative homology group $H_1(\Omega,\partial\Omega;\mathbf{R})$, and is hence a finite-dimensional vector space, with dimension equal to the number of components of $\partial\Omega$ minus the number of components of Ω.

The definition of the subspace GG of *grounded gradients*, which consists of all V in $\mathrm{VF}(\Omega)$ such that

$$V = \nabla\varphi,\ \varphi|_{\partial\Omega} = 0,$$

is self-explanatory.

We refer the reader to **[7]** for a thorough treatment of the Hodge Decomposition Theorem and a variety of applications to boundary value problems for vector fields.

CHAPTER 3

Axisymmetric Vector Fields

1. Introduction

The last chapter introduced two fundamental ideas of our story:

1. Energy–minimizing vector fields of fixed helicity are eigenfields of the modified Biot-Savart operator BS' of largest eigenvalue.
2. Curl is a left inverse for BS'; hence, finding eigenfields of BS' can be reduced to solving a system of partial differential equations.

We want to add the additional assumption of axial symmetry to this story, and prove that these ideas still work.

Let Ω be an axisymmetric solid torus in 3-space, with smooth boundary. In the last chapter, we defined $\mathrm{K}(\Omega)$ to be the space of smooth, divergence-free vector fields defined on Ω and tangent to $\partial\Omega$. We define the subspace $\mathrm{SK}(\Omega)$ of $\mathrm{K}(\Omega)$ to be the space of rotationally symmetric vector fields in $\mathrm{K}(\Omega)$.

We will first prove:

THEOREM 4. *Let V be a vector field in the L^2 closure of $\mathrm{SK}(\Omega)$ which has minimum energy among all such vector fields of fixed positive helicity. Then V is a smooth eigenfield of the modified Biot-Savart operator BS' with the largest eigenvalue among all such vector fields.*

This theorem does not immediately follow from the corresponding result for arbitrary vector fields! After all, we have no guarantee that the largest eigenvalue of BS' on $\mathrm{SK}(\Omega)$ is equal to the largest eigenvalue of BS' on the larger space $\mathrm{K}(\Omega)$, even though the domain Ω is rotationally symmetric. For instance, Gimblett, Taylor, Turner and Hall [**10**] have investigated the eigenfields of curl on a cylindrical "soup can" of radius r and height h, and found that the first eigenfields of curl were axisymmetric only for h/r less than about 1.67.

We will then deduce that the image of BS', when restricted to $\mathrm{SK}(\Omega)$, is the space of symmetric Amperian knots:

$$\mathrm{SAK}(\Omega) = \left\{ V \in \mathrm{SK}(\Omega) \;\middle|\; \begin{array}{c} \int_C V \cdot \mathrm{ds} = 0 \text{ for every curve } C \text{ in } \partial\Omega \\ \text{which bounds outside } \Omega \end{array} \right\}.$$

We can then trade the integral operator BS' for the differential operator curl:

COROLLARY 1. *Let V be the eigenfield of curl in $\mathrm{SAK}(\Omega)$ with least positive eigenvalue. Then V is the vector field in the L^2 closure of $\mathrm{SK}(\Omega)$ with least energy among all such vector fields of fixed positive helicity.*

To use these results later, we will need to write the space $\mathrm{SAK}(\Omega)$ in a different language.

PROPOSITION 1. *An axisymmetric vector field* V *in* $\mathrm{SK}(\Omega)$, *written in cylindrical coordinates as*

$$V = u(r,z)\hat{r} + v(r,z)\hat{\phi} + w(r,z)\hat{z},$$

lies in $\mathrm{SAK}(\Omega)$ *if and only if* $v(r,z) = 0$ *on* $\partial\Omega$.

2. Outline of Proof of Theorem 4

In Chapter 2, Section 7, we proved that

$$\mathrm{H}(V) = \langle V, \mathrm{BS}'(V)\rangle .$$

Later, we proved Theorem 2, which stated that BS' was a compact, self-adjoint operator on the L^2 closure of $\mathrm{K}(\Omega)$. Using these facts, the *Spectral Theorem* told us that the eigenfields of BS' of largest eigenvalue maximized the "Rayleigh quotient" of helicity and energy,

$$\frac{\mathrm{H}(V)}{\mathrm{E}(V)} = \frac{\langle V, \mathrm{BS}'(V)\rangle}{\langle V, V\rangle},$$

among all vector fields in the L^2 closure of $\mathrm{K}(\Omega)$. Hence, the vector fields of fixed helicity with the least energy among all vector fields were guaranteed to be the eigenfields of BS' of largest eigenvalue.

To prove the corresponding result on $\mathrm{SK}(\Omega)$, we need only check that BS' maps $\mathrm{SK}(\Omega)$ to $\mathrm{SK}(\Omega)$. Since $\mathrm{SK}(\Omega) \subset \mathrm{K}(\Omega)$, it will follow from Theorem 2 that BS' is a compact self-adjoint operator on the L^2 closure of $\mathrm{SK}(\Omega)$, and the spectral theorem will then give us Theorem 4.

3. The Image of BS' on $\mathrm{SK}(\Omega)$ is in $\mathrm{SK}(\Omega)$

PROPOSITION 2. BS' *maps* $\mathrm{SK}(\Omega)$ *into* $\mathrm{SK}(\Omega)$.

We start by fixing some $V \in \mathrm{SK}(\Omega)$. We wish to prove that $\mathrm{BS}(V)$ is also rotationally symmetric. This follows directly from examining the integral which defines $\mathrm{BS}(V)$:

$$\mathrm{BS}(V)(p) = \frac{1}{4\pi}\int_{\Omega} V(q) \times \frac{p-q}{|p-q|^3}\,\mathrm{dVol}_q .$$

Since the integrand commutes with any rigid motion carrying p and q to new locations, the integral must be commute with such motions as well. Writing out the argument in detail is straightforward.

Now $\mathrm{BS}'(V)$ is the L^2 orthgonal projection of $\mathrm{BS}(V)$ into $\mathrm{K}(\Omega)$. Consulting the Hodge Theorem of Chapter 2, we see that

$$\mathrm{BS}'(V) = \mathrm{BS}(V) + \nabla f,$$

for some function f on Ω. Since $\mathrm{BS}'(V)$ and $\mathrm{BS}(V)$ are divergence-free, the function f is harmonic. Further, since $\mathrm{BS}'(V)$ is tangent to the boundary of Ω,

$$\frac{\partial f}{\partial n} = \nabla f \cdot n = \mathrm{BS}(V) \cdot n$$

on $\partial\Omega$.

We have just shown that f solves a Neumann problem on Ω, with the boundary data $\partial f/\partial n = \mathrm{BS}(V)\cdot n$. Since $\mathrm{BS}(V)$ is axisymmetric the boundary data $\mathrm{BS}(V)\cdot n$ of this Neumann problem is axisymmetric. Since the solution of a Neumann problem is unique, this implies that f is axisymmetric as well.

We have just shown that ∇f and $\mathrm{BS}(V)$ are both axisymmetric. This proves that $\mathrm{BS}'(V)$ is axisymmetric, completing the proof of the Proposition.

4. The Image of BS' on $\mathrm{SK}(\Omega)$ is $\mathrm{SAK}(\Omega)$

In *Upper Bounds for The Writhing of Knots and the Helicity of Vector Fields* [**4**], we proved that the image of BS' on $\mathrm{K}(\Omega)$ was precisely the space of *Amperian* knots

$$\mathrm{AK}(\Omega) = \left\{ V \in \mathrm{K}(\Omega) \;\middle|\; \begin{array}{c} \int_C V\cdot \mathrm{ds} = 0 \text{ for every curve } C \text{ in } \partial\Omega \\ \text{which bounds outside } \Omega \end{array} \right\}.$$

We also proved that curl was the inverse for BS' on $\mathrm{AK}(\Omega)$.

We claim that the image of BS' on $\mathrm{SK}(\Omega)$ is the space of symmetric Amperian knots:

$$\mathrm{SAK}(\Omega) = \left\{ V \in \mathrm{SK}(\Omega) \;\middle|\; \begin{array}{c} \int_C V\cdot \mathrm{ds} = 0 \text{ for every curve } C \text{ in } \partial\Omega \\ \text{which bounds outside } \Omega \end{array} \right\}.$$

We already know that if $V = \mathrm{BS}(W)$, and $W \in \mathrm{SK}(\Omega)$, then $V \in \mathrm{SAK}(\Omega)$. Suppose $V \in \mathrm{SAK}(\Omega)$. We claim that $V = \mathrm{BS}(W)$, for some $W \in \mathrm{SK}(\Omega)$. Since curl is the inverse for BS' on $\mathrm{AK}(\Omega)$,

$$V = \mathrm{BS}'(\nabla \times V),$$

and $\nabla\times V \in \mathrm{K}(\Omega)$. We already know that V is axisymmetric. Thus $\nabla\times V$ is axisymmetric, and hence $\nabla\times V \in \mathrm{SK}(\Omega)$, proving our claim. This shows that the image of BS' on $\mathrm{SK}(\Omega)$ is exactly $\mathrm{SAK}(\Omega)$.

5. Proof of Proposition 1

We now want to show that an axisymmetric vector field

$$V = u(r,z)\hat{r} + v(r,z)\hat{\phi} + w(r,z)\hat{z}$$

in $\mathrm{SK}(\Omega)$ is in $\mathrm{SAK}(\Omega)$ if and only if $v = 0$ on $\partial\Omega$. To start, we observe that $v(r,z) = V\cdot\hat{\phi}$. Thus, $v(r,z) = 0$ on $\partial\Omega$ if and only if $V\cdot\hat{\phi} = 0$ on $\partial\Omega$.

Suppose $v = 0$ on $\partial\Omega$. Fix a curve $C \subset \partial\Omega$ which bounds outside Ω. We must show that

$$\int_C V\cdot ds = 0.$$

Since V is rotationally symmetric, it is clear that V is the gradient of a multivalued function on $\partial\Omega$, and in particular that V is curl-free on $\partial\Omega$.

The deRham theorem allows us to write any curl-free vector field V on $\partial\Omega$ in the form

$$V = \nabla f + V',$$

where V' is a harmonic (or curl and divergence-free) vector field representing a cohomology class in $H^1(\partial\Omega, \mathbf{R})$. We know that $H_1(\partial\Omega,\mathbf{R}) \cong \mathbf{R}\times\mathbf{R}$, where the homology generators are represented by the longitude C_1 and meridian C_2 shown below.

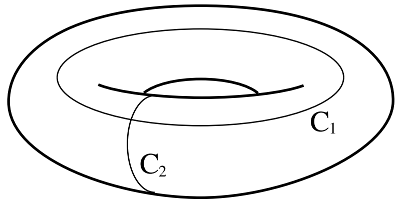


Figure 15: Homology generators for a torus.

Every curve on $\partial\Omega$ is homologous to a linear combination of C_1 and C_2. Since C bounds outside Ω, C is homologous to some scalar multiple of C_1 alone. Suppose $C \sim kC_1$. We always have

$$\int_C \nabla f \cdot ds = k \int_{C_1} \nabla f \cdot ds = 0.$$

Further, since $C \sim kC_1$ and $\nabla \times V' = 0$ on $\partial\Omega$,

$$\int_C V' \cdot ds = k \int_{C_1} V' \cdot ds.$$

Thus

$$\int_C V \cdot ds = k \int_{C_1} V \cdot ds.$$

But C_1 is a longitude, so C_1' is in the $\hat{\phi}$ direction on $\partial\Omega$. In particular, since $V \cdot \hat{\phi}$ vanishes on $\partial\Omega$, we have

$$\int_{C_1} V \cdot ds = 0.$$

This proves that V is in $\mathrm{SAK}(\Omega)$.

We now argue in the other direction. Suppose that V is in $\mathrm{SAK}(\Omega)$. We need to prove that $V \cdot \phi$ is 0 on $\partial\Omega$.

Fix some p on $\partial\Omega$, and consider the curve C generated by rotating p around the z-axis. As below, C bounds a surface outside Ω.

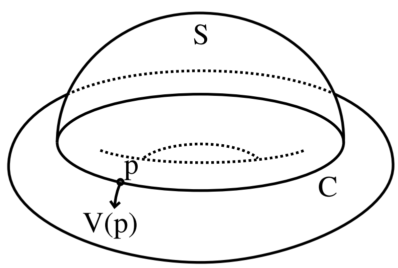


Figure 16: The surface bounded by C.

Thus, since $V \in \mathrm{SAK}(\Omega)$,

$$\int_C V \cdot \mathrm{ds} = 0.$$

Now C is a circle of revolution, so ds is a positive multiple of $\hat{\phi}$. Further, V is axisymmetric, so the inner product $V \cdot \hat{\phi}$ is constant along C. These facts imply that $V \cdot \hat{\phi} = 0$ at p.

This completes the proof of the Proposition, and the Chapter.

CHAPTER 4

Reduction to a Planar Problem

1. Introduction

In the last chapter, we proved that the energy-minimizing vector fields in $\mathrm{SK}(\Omega)$ of fixed positive helicity were the eigenfields of BS' with largest positive eigenvalue. We then defined the space $\mathrm{SAK}(\Omega) \subset \mathrm{SK}(\Omega)$, and proved that curl was the inverse of BS' on $\mathrm{SAK}(\Omega)$. This showed that the eigenfields of BS' of largest positive eigenvalue in $\mathrm{SK}(\Omega)$ were the eigenfields of curl with least positive eigenvalue in $\mathrm{SAK}(\Omega)$.

The goal of this chapter will be to study the eigenfield problem for curl on $\mathrm{SAK}(\Omega)$ as a system of first-order partial differential equations on the 3-dimensional domain Ω, and to reduce this problem to a *single* second-order partial differential equation on a 2-dimensional cross section of Ω.

We will start with a correspondence between functions and vector fields.

PROPOSITION 3. *Let Ω be a rotationally symmetric solid torus with smooth boundary. Suppose that the vector field*

$$V = u(r,z)\hat{r} + v(r,z)\hat{\phi} + w(r,z)\hat{z}$$

is an eigenfield of curl in $\mathrm{SAK}(\Omega)$ with eigenvalue λ. Let $C(\Omega)$ be a cross-section of Ω in an r-z plane.

Then the function $rv(r,z)$ is an eigenfunction of the operator

$$L[f] = f_{rr} + f_{zz} - \frac{1}{r} f_r,$$

on $C(\Omega)$, with Dirichlet boundary conditions, and eigenvalue $-\lambda^2$.

The converse is provided by

PROPOSITION 4. *Let $C(\Omega)$ be the cross-section of a solid torus of revolution Ω. Suppose that the function f is an eigenfunction of L on $C(\Omega)$, with Dirichlet boundary conditions, and eigenvalue $-\lambda^2$. Then the vector field*

$$V = \frac{1}{\lambda r}\left[-f_z\hat{r} + \lambda f\hat{\phi} + f_r\hat{z}\right] = \frac{1}{\lambda r}(\nabla f \times \hat{\phi}) + \frac{f}{r}\hat{\phi}.$$

is an eigenfield of curl in $\mathrm{SAK}(\Omega)$ with eigenvalue λ.

These propositions will make it easy to show

COROLLARY 2. *The spectrum of curl on $\mathrm{SAK}(\Omega)$ is symmetric about zero. Using the correspondence above, we see that λ and $-\lambda$ are eigenvalues of curl on $\mathrm{SAK}(\Omega)$ if and only if $-\lambda^2$ is an eigenvalue of L on $C(\Omega)$ with Dirichlet boundary conditions.*

This will complete the Chapter.

2. The system of partial differential equations

Suppose that

$$V = u(r, \phi, z)\hat{r} + v(r, \phi, z)\hat{\phi} + w(r, \phi, z)\hat{z}.$$

Then

$$\nabla \times V = \left[\frac{1}{r}w_\phi - v_z\right]\hat{r} + [u_z - w_r]\,\hat{\phi} + \frac{1}{r}\left[(rv)_r - u_\phi\right]\hat{z}.$$

If V is axisymmetric,

$$V = u(r, z)\hat{r} + v(r, z)\hat{\phi} + w(r, z)\hat{z},$$

and

$$\nabla \times V = -v_z\hat{r} + [u_z - w_r]\hat{\phi} + \frac{1}{r}(rv)_r\hat{z}.$$

Using this, we can rewrite the equation $\nabla \times V = \lambda V$ as a system of three first-order partial differential equations:

$$-v_z = \lambda u. \quad u_z - w_r = \lambda v. \quad \frac{1}{r}(rv)_r = \lambda w.$$

3. Proof of Proposition 3

We are now ready to prove Proposition 3. Suppose that V is an eigenfield of curl in $\mathrm{SAK}(\Omega)$ with eigenvalue λ. Since V is axisymmetric, if

$$V = u(r, z)\hat{r} + v(r, z)\hat{\phi} + w(r, z)\hat{z},$$

then u, v, and w satisfy the system of first order PDEs above.

We claim that this implies rv is an eigenfunction of L with eigenvalue $-\lambda^2$. To prove it, we start by writing

$$\begin{aligned}
L[rv] &= (rv)_{zz} + (rv)_{rr} - \frac{1}{r}(rv)_r \\
&= r(v_{zz}) + r\left(\frac{1}{r}(rv)_{rr} - \frac{1}{r^2}(rv)_r\right) \\
&= r(v_z)_z + r\left(\frac{1}{r}(rv)_r\right)_r.
\end{aligned}$$

So far, we've just used the definition of L and some algebraic manipulations. Using the first and third of our system of PDEs,

$$\begin{aligned}
L[rv] &= r(-\lambda u)_z + r(\lambda w)_r \\
&= -r\lambda(u_z - w_r).
\end{aligned}$$

Using the second equation in our system, we have

$$\begin{aligned}
L[rv] &= -r\lambda(\lambda v) \\
&= -\lambda^2(rv),
\end{aligned}$$

proving the claim.

We have proved that rv is an eigenfunction of L on a cross-section $C(\Omega)$ of Ω. Since V is in $\mathrm{SAK}(\Omega)$, v vanishes on $\partial\Omega$ by Proposition 1 of Chapter 3 and rv obeys Dirichlet boundary conditions.

4. Proof of Proposition 4

We now assume that we have an eigenfunction f of L with eigenvalue $-\lambda^2$ on the cross-section $C(\Omega)$ of a solid torus of revolution Ω. We want to prove that the vector field

$$V = \frac{1}{\lambda r}\left[-f_z\hat{r} + \lambda f\hat{\phi} + f_r\hat{z}\right]$$

is an eigenfield of curl in $\mathrm{SAK}(\Omega)$ with eigenvalue λ.

We first check that V is an eigenfield of curl. Clearly, V is rotationally symmetric. Thus, if $V = u\hat{r} + v\hat{\phi} + w\hat{z}$, the equation $\nabla \times V = \lambda V$ can be rewritten as the system of partial differential equations

$$\lambda u = -v_z, \quad \lambda v = u_z - w_r, \quad \lambda w = \frac{1}{r}(rv)_r.$$

In our case,

$$u = \frac{-1}{\lambda r}f_z, \quad v = \frac{1}{r}f, \quad w = \frac{1}{\lambda r}f_r.$$

It is easy to check that $\lambda u = -v_z$, and that $\lambda w = (1/r)(rv)_r$. To check the second equation in our system of PDE's, we write down

$$\begin{aligned} u_z - w_r &= -\frac{1}{\lambda r}f_{zz} - \frac{1}{\lambda r}f_{rr} + \frac{1}{\lambda r^2}f_r \\ &= -\frac{1}{\lambda r}\left[f_{zz} + f_{rr} - \frac{1}{r}f_r\right] \\ &= -\frac{1}{\lambda r}L[f] \\ &= \lambda\frac{f}{r} \\ &= \lambda v. \end{aligned}$$

This proves that V, as defined above, is an eigenfield of curl with the correct eigenvalue.

We must still check that V is in $\mathrm{SAK}(\Omega)$. By Proposition 1 of Chapter 3, we must prove three things:

LEMMA 1. *V is divergence-free.*

To check that V is divergence-free, we observe that

$$\begin{aligned} \nabla \cdot V &= \nabla \cdot \frac{1}{\lambda}\nabla \times V \\ &= \frac{1}{\lambda}\nabla \cdot \nabla \times V \\ &= 0. \end{aligned}$$

LEMMA 2. *V is tangent to $\partial\Omega$.*

Fix a point p on $\partial\Omega$, and the normal vector n to $\partial\Omega$ at p. We must show that $V \cdot n = 0$. By assumption, f obeys Dirichlet boundary conditions on Ω, so f is identically zero on $\partial\Omega$. Thus, ∇f is a scalar multiple of n. Suppose $\nabla f = kn$. Then

$$k(V \cdot n) = \nabla f \cdot V = uf_r + vf_\phi + wf_z.$$

Since f is axisymmetric, $f_\phi = 0$. Substituting in the definitions of u and w, we have

$$\nabla f \cdot V = -\frac{f_z f_r}{\lambda r} + \frac{f_r f_z}{\lambda r} = 0.$$

Thus $V \cdot n$ is 0 or ∇f is 0. If $V \cdot n$ is 0, we're done. If $\nabla f = 0$, then $f_r = 0$ and $f_z = 0$, so

$$V = \frac{1}{r} f \hat{\phi}.$$

Since Ω is rotationally symmetric, this implies $V \cdot n = 0$, proving that V is tangent to $\partial\Omega$.

LEMMA 3. *$v = 0$ on $\partial\Omega$.*

By assumption, f obeys Dirichlet boundary conditions, so $f = 0$ on $\partial\Omega$. Thus, since

$$v = \frac{1}{r} f,$$

it is clear that $v = 0$ on $\partial\Omega$.

These lemmae tell us that V is in $\mathrm{SAK}(\Omega)$, completing the proof of Proposition 4.

CHAPTER 5

The Eigenfunctions of L

1. Introduction

In the last chapter, we proved that the eigenfield problem for curl on $\mathrm{SAK}(\Omega)$ was equivalent to the eigenfunction problem for the second-order differential operator

$$L[f] = f_{rr} + f_{zz} - \frac{1}{r} f_r,$$

on a cross-section of Ω in an r-z plane, with Dirichlet boundary conditions.

In this chapter, our goal will be to study the eigenfunctions of L. It will be easy to check that L is an *elliptic* differential operator, and that L obeys a maximum principle. The standard theory of elliptic partial differential equations will then tell us a number of facts about the *first* eigenfunction f_1 of L. In particular, f_1 must be strictly positive inside Ω, and the normal derivative of f must not vanish at any point on $\partial\Omega$.

It is at this point where our story differentiates "tokamak" domains, or solid tori, from "spheromak" domains, or solid balls. The difference is that axisymmetric solid balls must intersect the axis of revolution. At that intersection, the term $(1/r)f_r$ of the equation $L[f] = -\lambda^2 f$ will have a singularity inside Ω. This will prevent us from reaching the conclusions above about the positivity of the first eigenfunction of L on the cross section of a "spheromak", and so stop us from extending the rest of our arguments to those domains.

On "tokamak" domains, however, these facts will make it easy to prove

THEOREM 5. *If $V = u(r,z)\hat{r} + v(r,z)\hat{\phi} + w(r,z)\hat{z}$ is the first eigenfield of curl in $\mathrm{SAK}(\Omega)$ on an axisymmetric solid torus Ω with smooth boundary, then $v(r,z)$ is strictly positive inside Ω.*

This property characterizes the first eigenfield:

THEOREM 6. *If $V = u(r,z)\hat{r} + v(r,z)\hat{\phi} + w(r,z)\hat{z}$ is any eigenfield of curl in $\mathrm{SAK}(\Omega)$ on an axisymmetric solid torus Ω with smooth boundary, and $v(r,z)$ is strictly positive inside Ω, then V is the* **first** *eigenfield of curl in $\mathrm{SAK}(\Omega)$.*

We do not need the assumption that $\partial\Omega$ is smooth in these theorems.

THEOREM 7. *Suppose Ω is a rotationally symmetric solid torus with smooth boundary. Then the first eigenfield of curl in $\mathrm{SAK}(\Omega)$ never vanishes on Ω or its boundary.*

Axisymmetric eigenfields of curl of larger eigenvalue may have zeros on Ω, but even then the zeros have a definite structure:

THEOREM 8. *Suppose $V = u(r,z)\hat{r} + v(r,z)\hat{\phi} + w(r,z)\hat{z}$ is an eigenfield of curl in $\mathrm{SAK}(\Omega)$. Then V vanishes only where the level set $rv = 0$ has a singularity, as in*

Figure 8. In this case, V vanishes along a circle. These "nodal circles" are isolated– no

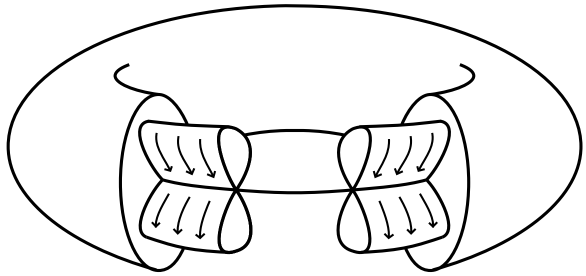


Figure 17: A nodal circle.

axisymmetric curl eigenfield vanishes on a two-dimensional surface.

This will complete the Chapter.

2. The operator L

We start with a Proposition about L.

PROPOSITION 5. *The second order differential operator L is a uniformly elliptic operator with bounded coefficients on the cross-section $C(\Omega)$. Further, L obeys a maximum principle on $C(\Omega)$.*

We begin with the operator L:

$$L[f] = f_{rr} + f_{zz} - \frac{1}{r} f_r.$$

Clearly, L is the Laplace operator in the (Euclidean) r-z plane, with a first-order correction term.

A general second order differential operator in the form

$$M[f] = \sum_{ij} a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_k b_k(x) \frac{\partial f}{\partial x_k} + c(x) f$$

is said to be *elliptic* if the matrix of coefficients $a_{ij}(x)$ is positive-definite at every point on the domain. That is, f is elliptic if, for any vector (x_1, x_2), at each point x in the domain, we have

$$\left| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right| > k(x) \left| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right|,$$

for some $k(x) > 0$. If $k(x) > \epsilon > 0$ at every point in the domain, we say that M is *uniformly elliptic*.

For the operator L, the matrix of coefficients a_{ij} is the identity matrix. Thus, it's clear that L is uniformly elliptic on the domain $C(\Omega)$.

An elliptic operator obeys a *maximum principle* on a domain if every function f with $M[f]$ identically zero acheives its maximum value on the boundary of the domain. Any elliptic operator in the form above obeys a maximum principle if the linear coefficient $c(x)$ is nonpositive. Since $c = 0$ for L, we know that L obeys a maximum principle on $C(\Omega)$.

All of the other coefficients of L are zero, except for the first–order coefficient $b_1 = -1/r$. Since Ω stays away from the z axis, this term is bounded, and so all of the coefficients of L are bounded on $C(\Omega)$. This completes the proof of the proposition.

3. Positivity of the first eigenfunction

It is a classical result that on a compact domain with smooth boundary, the first eigenfunction of a uniformly elliptic differential operator which obeys a maximum principle and has bounded coefficients must be positive inside the domain, if Dirichlet boundary conditions are imposed. Further, the first eigenfunction is the only eigenfunction with this property.

This result has been improved in more modern papers, to apply to domains whose boundaries are not smooth. The most modern result in this direction is the proof of Berestycki, Nirenberg, and Varadhan in 1993 [**2**] that first eigenfunctions of such operators exist and are positive if the domain is *any* bounded open subset of $\mathbf{R}^n$, though the meaning of "Dirichlet boundary conditions" must be reinterpreted for very strange domains. We will use this result only for domains with piecewise smooth boundaries and corners of positive angle, as below: where the meaning of "Dirichlet boundary conditions" remains

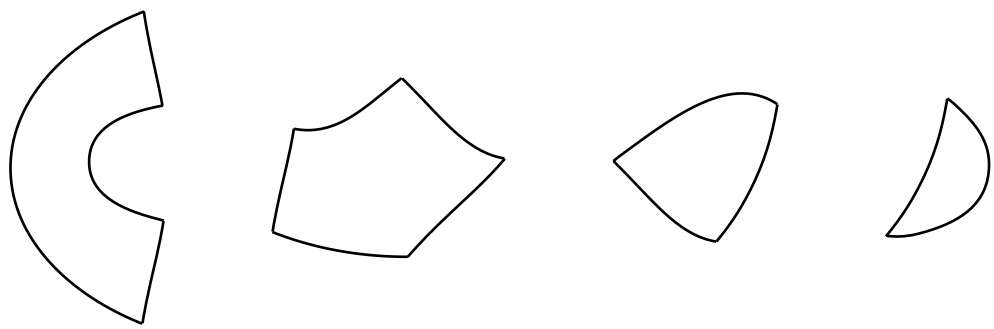


Figure 18: Examples of domains satisfying our hypotheses.

unchanged.

Similarly, it is a classical result that the first eigenvalue of such an operator must be simple. Berestycki, Nirenberg and Varadhan prove simplicity of the first eigenvalue on arbitrary domains in the same paper, and we cite them again for this result.

This allows us to conclude:

PROPOSITION 6. *The first eigenfunction f of L on the cross-section $C(\Omega)$ of a rotationally symmetric solid torus in 3-space, with Dirichlet boundary conditions, must obey*

$$f(r, z) > 0,$$

inside $C(\Omega)$. The function f is the only eigenfunction of L with this property. And the first eigenvalue of L is simple.

4. The normal derivatives of the first eigenfunction.

The fact that the normal derivative of the first eigenfunction of an elliptic operator which obeys a maximum principle never vanishes is known as the Hopf boundary point lemma.

Even for domains with corners, the Hopf lemma is classical, and the version required here can be found as Theorem 7 on Page 65 of Protter and Weinberger's beautiful textbook *Maximum Principles in Differential Equations* [**15**].

We have concluded:

PROPOSITION 7. *The first eigenfunction f of L on the cross-section $C(\Omega)$ of a rotationally symmetric solid torus in 3-space, with Dirichlet boundary conditions, must have*

$$\frac{\partial f}{\partial n} > 0,$$

where n is the inward-pointing normal vector on $\partial C(\Omega)$, wherever n is well-defined.

5. Proof of Theorem 5

We now prove that if

$$V = u(r,z)\hat{r} + v(r,z)\hat{\phi} + w(r,z)\hat{z}$$

is an eigenfield of curl in $\mathrm{SAK}(\Omega)$ of least positive eigenvalue λ, then $v(r,z) > 0$ inside Ω.

The correspondance betweeen eigenfields of curl and eigenfunctions of L expressed in Propositions 3 and 4 of Chapter 4 tells us that $-\lambda^2$ is the first eigenvalue of L with Dirichlet boundary conditions on the cross section $C(\Omega)$ of Ω. Further, the corresponding eigenfunction is the function rv. Proposition 6 tells us that $rv > 0$ inside Ω, and proves the Theorem. The proof of the converse— Theorem 6— also comes directly from Proposition 6.

6. Proof of Theorem 7

We now prove that the first eigenfield V of curl on an axisymmetric solid torus Ω never vanishes on Ω or its boundary.

The fact that V never vanishes inside Ω comes from the last theorem— after all, if

$$V = u(r,z)\hat{r} + v(r,z)\hat{\phi} + w(r,z)\hat{z},$$

and $v > 0$ at some point, it is clear that V does not vanish at that point!

To prove that V never vanishes on the boundary, we recall the specifics of our correspondence between the vector field V and the function rv. Proposition 4 of Chapter 4 tells us that

$$V = \frac{1}{\lambda r}(\nabla(rv) \times \hat{\phi}) + v\hat{\phi}.$$

On the boundary, v vanishes. However, since rv is the first eigenfunction of L on the cross-section, Proposition 7 tells us that the normal derivative of rv is positive on $\partial\Omega$. This means that ∇f cannot vanish on $\partial\Omega$, and so V cannot vanish on $\partial\Omega$.

7. The Structure of Zeros of Other Eigenfields

We now want to consider the structure of zeros of other eigenfields of curl in $\mathrm{SAK}(\Omega)$, and prove Theorem 8. In particular, we will show that the nodal sets of these eigenfields are always one-dimensional. Suppose we have some such eigenfield

$$V = u(r,z)\hat{r} + v(r,z)\hat{\phi} + w(r,z)\hat{z}.$$

Let's think about the level set $v(r, z) = 0$. Clearly, this is also a level set of the function $rv(r, z)$, and clearly, if there are any zeros of V, they must lie on this set.

Looking forward to Chapter 6, we know that the level set $rv = 0$ is a finite collection of smoothly immersed circles in the plane, plus the boundary of $C(\Omega)$ and a collection of arcs joining points on $\partial\Omega$, all intersecting each other transversely, as below: These curves carve

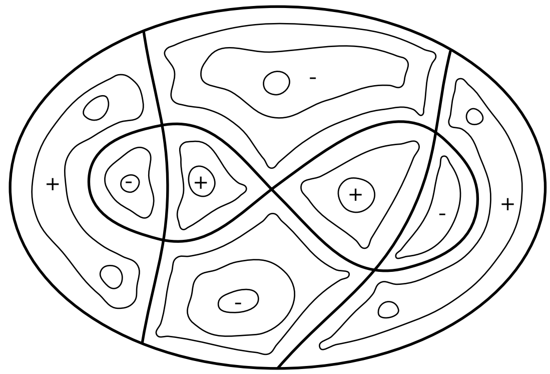


Figure 19: The nodal set divides $C(\Omega)$ into subdomains.

$C(\Omega)$ into a collection of subdomains, each with piecewise smooth boundary. On each, rv is strictly positive (or negative), and on the boundary of each subdomain, rv vanishes. So rv obeys Dirichlet boundary conditions on each subdomain, and by Proposition 6, rv (or $-rv$) is the *first* eigenfunction of L on each subdomain.

Proposition 7 then tells us that $\partial(rv)/\partial n$ does not vanish on the level set $rv = 0$, wherever the normal vector n to that set is well-defined. This tells us that $\nabla(rv)$ can vanish only where curves in the level set $rv = 0$ intersect, and it is clear that $\nabla(rv)$ *must* vanish at these intersections. To see that V vanishes at these intersections, and nowhere else, just recall that

$$V = \frac{1}{\lambda r}(\nabla(rv) \times \hat{\phi}) + v\hat{\phi}.$$

Since these intersections must be isolated points, we have proved Theorem 8.

CHAPTER 6

The Integral Surfaces of V

1. Introduction

In the last chapter, we focused on the eigenfunctions of L. Using the theory of elliptic operators, we were able to show that the first eigenfield of curl in $\mathrm{SAK}(\Omega)$ never vanishes on Ω or $\partial\Omega$, and that V always has a nonzero (by convention, positive) component in the $\hat{\phi}$ direction inside Ω.

In this chapter, we first prove that every eigenfield of curl in $\mathrm{SAK}(\Omega)$ is tangent to a family of integral surfaces which foliate Ω. Then, we investigate the structure of these surfaces. The fact that these are level surfaces of an analytic function will give us some information about the surfaces and their intersections. For the *first* eigenfield, a maximum principle argument will allow us to derive much stronger results.

The first theorem must be:

THEOREM 9. *Using the cylindrical coordinates r, ϕ, and z, every axisymmetric eigenfield*

$$V = u(r,z)\hat{r} + v(r,z)\hat{\phi} + w(r,z)\hat{z}$$

of curl is tangent to the level surfaces of the function rv, as shown in Figure 9.

Next, we will invoke the fact that rv is an analytic function inside the domain Ω to prove

PROPOSITION 8. *The level surfaces of the analytic, rotationally symmetric function rv foliate Ω. Each connected level surface with $rv > 0$ is a smoothly immersed collection of tori.*

We will then turn to the structure of the level curves of the *first* eigenfunction of L. Most of our results will follow from the following monotonicity principle:

PROPOSITION 9. *Let γ be any closed curve in $C(\Omega)$ enclosing a region R. Let f be the first eigenfunction of L on $C(\Omega)$, with Dirichlet boundary conditions.*
If $f \geq b$ on γ, then $f > b$ inside R.

Using this principle, we will be able to obtain more information about the integral surfaces of the first eigenfield. We'll say that a level curve of rv is *singular* if $\nabla(rv)$ vanishes anywhere on the curve. From this definition, we know that singular level curves are isolated.

THEOREM 10. *Let V be the first eigenfield of curl in $\mathrm{SAK}(\Omega)$. Every connected, non-singular, level curve of the function rv is a smoothly embedded circle, and every connected level curve of rv is a smoothly immersed circle.*

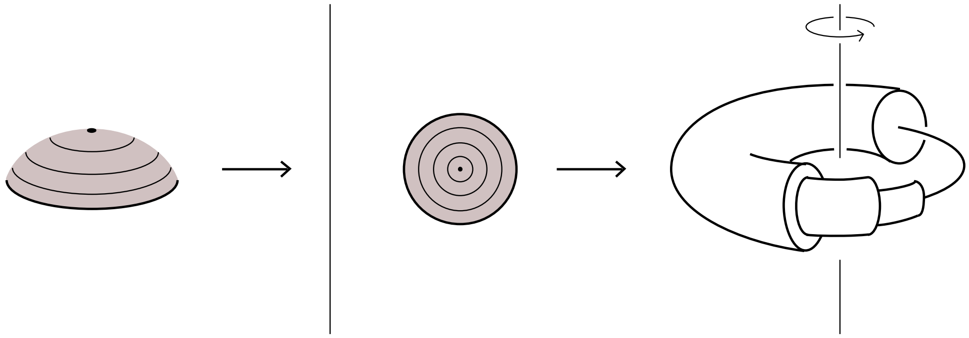


Figure 20: The level curves of rv become integral surfaces of V.

Thus, every connected, nonsingular, integral surface of V is a smoothly embedded torus and every connected, singular integral surface of V, except for $\partial\Omega$ is a smoothly immersed torus.

Given an immersed circle C in the plane, we can see that C encloses a collection of bounded open domains, as below left. We can construct the graph $G(C)$ of this circle by assigning a vertex to each open domain, and joining these vertices with edges when the corresponding regions share boundary points, as below right.

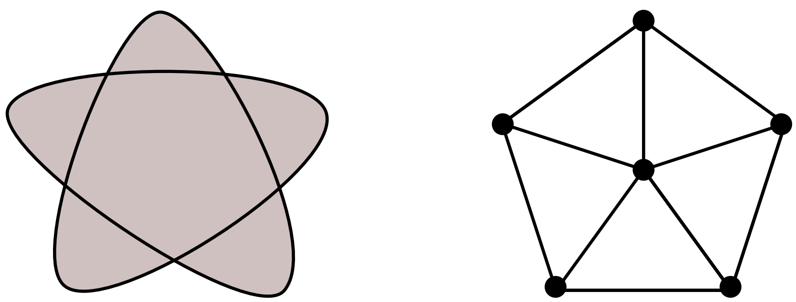


Figure 21: An immersed circle, the domains it bounds, and their graph.

THEOREM 11. *Let V be the first eigenfield of curl in $\mathrm{SAK}(\Omega)$, and let C be the cross section of a singular integral surface of V. Then the graph $G(C)$ is a tree.*

The content of this theorem is summarized by Figure 1.

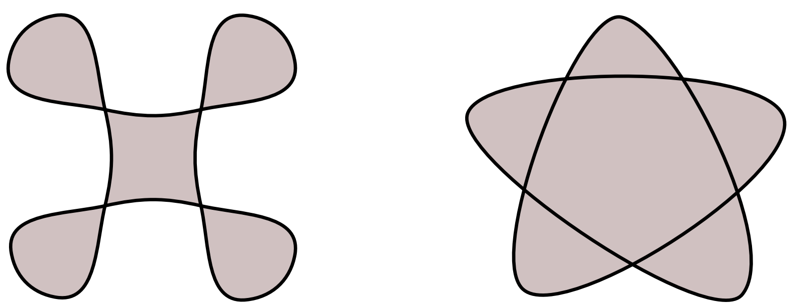


Figure 22: Permitted and forbidden cross-sections for singular integral surfaces.

Given a tree, we should be able to find a planar domain on which the first eigenfunction of L has a singular level curve C whose graph $G(C)$ is the given tree. For example, the domain at left of Figure 1 should generate a singular level curve with the tree at the right of that Figure.

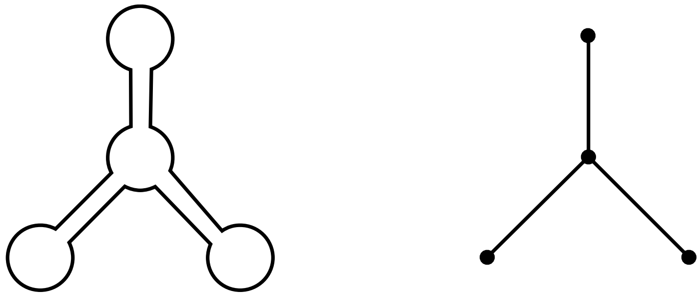


Figure 23: A domain with a desired singular leaf.

So this is the strongest structure theorem for singular integral surfaces which can be expected, without additional assumptions about the domain. On the axisymmetric solid torus of circular-cross section, for instance, we saw no singular integral surfaces of the first eigenfield of curl in SAK. A natural guess is

CONJECTURE 1. *The first eigenfield of curl on any solid torus with convex cross-section has no singular integral surfaces.*

Numerical evidence strongly supports this conjecture. I cannot yet prove this conjecture, but I can prove an improved structure theorem for the singular integral surfaces of V if the failure of the cross-section to be convex is controlled. The proof of this theorem is a modification of a similar argument of Payne [**14**].

THEOREM 12. *Let $C(\Omega)$ be a cross-section in an r-z plane of the axisymmetric solid torus Ω. Suppose $C(\Omega)$ has a flip symmetry over the r axis, and that the normal vector to $\partial C(\Omega)$ points in the $\hat{r}$ direction only where $\partial C(\Omega)$ crosses the r-axis, as below. Let V*

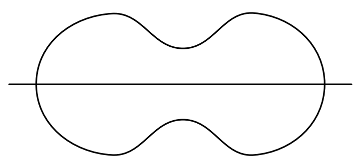


Figure 24: Cross-section of a domain satisfying the hypothesis of Theorem 12.

be the first eigenfield of curl in $\mathrm{SAK}(\Omega)$. Then the cross-section of every singular integral surface is one of the "multiple eights" below.

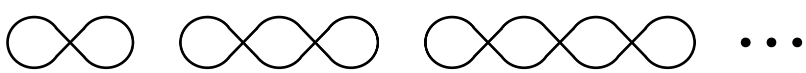


Figure 25: Possible cross-sections for singular integral surfaces.

2. Introduction to the Proof of Theorem 9

We now want to prove that every eigenfield

$$V = u(r,z)\hat{r} + v(r,z)\hat{\phi} + w(r,z)\hat{z}$$

of curl in $\mathrm{SAK}(\Omega)$ is tangent to the level surfaces of the function rv. We'll give two proofs of this proposition: one from the standpoint of our correspondence between eigenfunctions of L and eigenfields of curl, and another from a more general perspective.

3. First Proof

Using our correspondence between eigenfunctions of L and eigenfields of curl, we know that V can be rewritten as

$$V = \frac{1}{\lambda r}(\nabla(rv) \times \hat{\phi}) + v\hat{\phi}.$$

From this,

$$V \cdot \nabla(rv) = v\hat{\phi} \cdot grad(rv).$$

Since rv is an axisymmetric function, $\nabla(rv) \cdot \hat{\phi} = 0$, and this proves that V is tangent to the level surfaces of rv.

4. Second Proof

Suppose that W is a Killing field, and V is a curl eigenfield invariant under the flow of W. We have

$$V \cdot \nabla(V \cdot W) = V \cdot [V \times (\nabla \times W) + W \times (\nabla \times V)] + V \cdot \nabla_V W + V \cdot \nabla_W V.$$

Now $V \cdot [V \times (\nabla \times W)]$ is clearly 0. Further, since $\nabla \times V = \lambda V$, we know that $V \cdot [W \times (\nabla \times V)]$ also vanishes. We are left with

$$V \cdot \nabla(V \cdot W) = V \cdot \nabla_V W + V \cdot \nabla_W V.$$

However, the Killing equation promises that for any vector fields X and Y, we have

$$X \cdot \nabla_Y W + Y \cdot \nabla_X W = 0.$$

It follows that $V \cdot \nabla_V W = 0$. Last, since V is invariant under W, it is clear that $\nabla_W V$ must be zero.

Now suppose that $W = r\hat{\phi}$ is the Killing field generating rotation around the z-axis. If

$$V = u(r,z)\hat{r} + v(r,z)\hat{\phi} + w(r,z)\hat{z},$$

then

$$V \cdot W = rv.$$

Since V is surely invariant under the action generated by W, and V is also a curl eigenfield, the argument above proves that $V \cdot \nabla(rv) = 0$. This completes the proof.

5. Proof of Proposition 8

We have shown that V is tangent to the level surfaces of the function rv. It follows from elliptic regularity, or the Cauchy-Kowalevski Theorem, that the function rv is analytic inside Ω (see, for instance, our argument in [**4**] for a proof that curl eigenfields are real analytic inside Ω). There is plenty of information available about the the level sets of analytic functions— what we need is given by the following Proposition, which is a restatement of material in [**14**].

PROPOSITION 10. *Suppose $f(r, z)$ is a non-constant analytic function on a region R of the r-z plane. Each level set $f(r, z) = c$ contained within R is a smoothly immersed collection of circles, as below:*

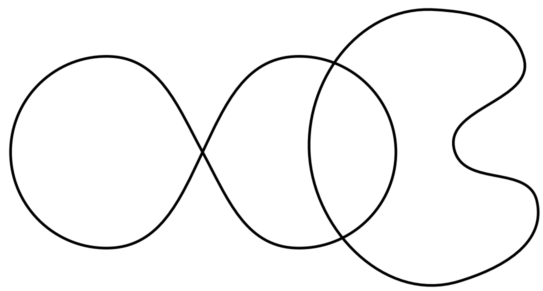


Figure 26: A typical singular level curve.

Notice that none of the level sets is two-dimensional: if f were to remain constant on a neighborhood of some point, then all derivatives of f would vanish at that point, and f would have to be constant everywhere by analyticity.

In the circumstances of Proposition 8, we know that rv is not constant on $C(\Omega)$. Further, since $rv = 0$ on the boundary of $C(\Omega)$, we know that each level set with $rv > 0$ is contained inside $C(\Omega)$. Thus Proposition 10 applies. Since the level surfaces of rv in Ω are generated by revolving the level curves of rv in $C(\Omega)$, this completes the proof.

6. The Monotonicity Principle for First Eigenfunctions

To prove Proposition 9, we start by assuming that there is some p inside the region R bounded by γ where $f \leq b$.

Since R is compact, f achieves a global minimum t on R. Since $t \leq b$, and $f \leq b$ at the interior point p, there must be some interior point q where $f(q) = t$.

Let's think about the derivatives of f at q. Clearly,

$$\nabla f(q) = 0.$$

Further, f is nondecreasing in every direction from q. Thus, at q,

$$f_{rr} \geq 0, \quad f_{zz} \geq 0.$$

This implies that $L[f]$ is nonnegative at q, since

$$L[f] = f_{rr} + f_{zz} - \frac{1}{r} f_r \geq 0.$$

On the other hand, f is strictly positive at p (by Proposition 6 of Chapter 5), and by assumption we have

$$L[f] = -\lambda^2 f.$$

Since λ is the first eigenvalue of L, it is nonzero by definition, and $L[f]$ is negative at q. The contradiction proves the result.

7. Proof of Theorem 10

To prove the Theorem, we focus our attention on the cross-section $C(\Omega)$ in an r-z plane. Proposition 10 has already given us a great deal of information about the structure of the level curves of rv inside $C(\Omega)$.

Pick a connected level curve $\gamma = \{p|rv(p) = c\}$ inside $C(\Omega)$. We prove that if γ is nonsingular, then γ is a smoothly embedded circle.

Let U be a small neighborhood of γ. If $\nabla(rv)$ never vanishes on γ, then c is a regular point of the map $rv : U \to \mathbf{R}$. Thus $(rv)^{-1}(c) \supset \gamma$ must be a one-dimensional manifold, or a collection of disjoint circles. Since γ is connected, γ must consist of a single circle, and the corresponding level surface of rv inside Ω must be a single embedded torus.

Proving that connected, *singular* integral surfaces are immersed tori requires the monotonicity principle of Proposition 9. Suppose a singular level curve $\gamma = \{p|rv = c\}$ of rv consists of two immersed circles, γ_1 and γ_2, meeting at some point, as below. It is clear

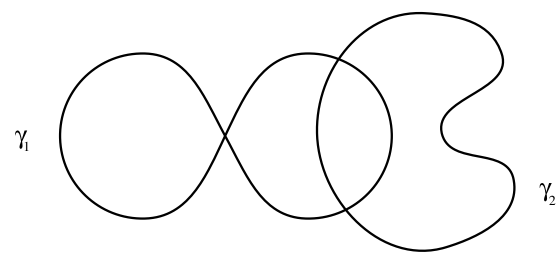


Figure 27: Immersed circles meeting at a point.

that part of γ_1 must lie inside the region bounded by γ_2. However, since $rv = c$ on γ_2, Proposition 9 tells us that $rv > c$ inside this region. Since $rv = c$ on γ_1, this implies that γ_1 and γ_2 cannot meet, which contradicts our assumption that γ was connected.

8. Proof of Theorem 11

Choose any singular integral surface $S = \{p|rv = c\}$ of the first eigenfield V of curl in $\mathrm{SAK}(\Omega)$. Since the surface is rotationally symmetric, we can reduce this picture to a cross-section $C(\Omega)$ and a singular level curve C of rv, as below. The graph $G(C)$ of C is shown at right. We can also choose points inside the regions bounded by C, and connect them by paths inside these regions to realize the graph $G(C)$ in the plane. If the graph $G(C)$ has a cycle, there will be a closed path γ connecting several of these points, as below: Every point p on γ is either on C, or inside a region bounded by C. In either case, Proposition 9 implies that $rv(p) \geq c$. Using the Proposition again, $rv > c$ on the region inside γ. But γ connects points in *different* regions bounded by C, so C must cut γ somewhere. This

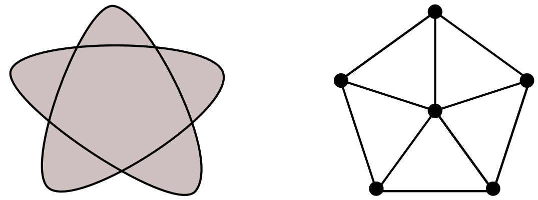


Figure 28: A singular level curve C and the graph $G(C)$.

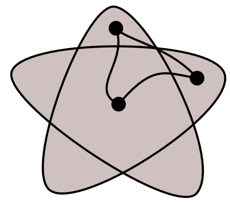


Figure 29: A cycle in the graph $G(C)$, realized in the plane.

means that points on C are within the region contained in γ, contradicting our conclusion that $rv > c$ on that region. Thus $G(C)$ has no cycles, which means that $G(C)$ is a tree.

9. Proof of Theorem 12

We will now prove the last theorem of the Chapter. Using our correspondence between eigenfields of curl and eigenfunctions of L, we must prove the following claim: Suppose we have a planar region R with a flip symmetry over the r-axis. Further, suppose that the boundary normal to ∂R points in the $\hat{r}$ direction only where ∂R crosses the r axis, as below:

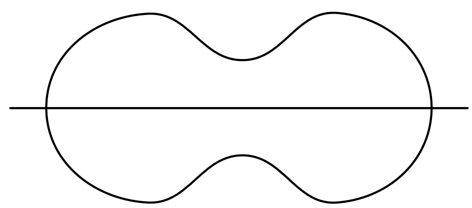


Figure 30: A cross-section satisfying the hypotheses of Theorem 12.

Let f be the first eigenfunction of L on R, with Dirichlet boundary conditions, and eigenvalue $-\lambda^2$. We claim that every local maximum and every saddle point of f lie on the r axis. This will imply that the singular integral surfaces of the first eigenfield of curl on the domain Ω with cross-section R have the form described in the conclusion of Theorem 12. (Of course, there are no local minima of f, by the monotonicity principle of Proposition 9.)

Suppose that a local max for f is located at p, off the r-axis. Consider the function f_z. It is easy to see that f_z is also an eigenfunction of L, with eigenvalue $-\lambda^2$. Clearly, ∇f vanishes at p, and so p is in the nodal set

$$C_z = \{p \in R | f_z(p) = 0\}.$$

Now f is analytic, so f_z is analytic, and the level set C_z consists of a collection of immersed circles, immersed arcs connecting points on the boundary of R, and isolated points.

We will show that p cannot be an isolated zero of f_z. Suppose p is on an immersed circle. Clearly, C_z encloses a region R' inside R, as below. Suppose p is on an immersed

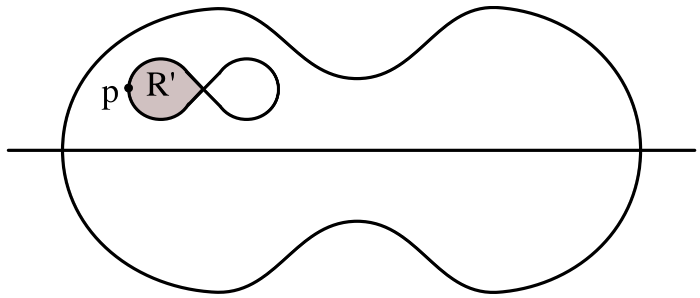


Figure 31: The case where p is on an immersed circle.

arc connecting points on ∂R. Using the Hopf boundary lemma of Chapter 5 and our assumptions about R, we know that the only points on ∂R in C_z are the two points where ∂R intersects the r axis. If the immersed arc closes, then C_z encloses a region R' inside R as below. Suppose the immersed arc joins our two points. We will show that f is flip

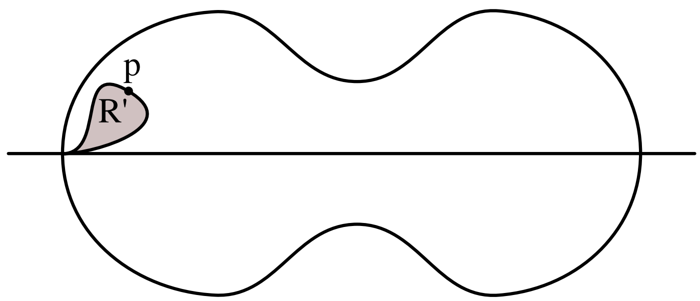


Figure 32: The first case where p is on an immersed arc.

symmetric over the r axis, so the portion of the r axis joining these points is also in C_z. Thus, as below, C_z encloses a region R' inside R. Since $f_z = 0$ on $\partial R'$, $f_z \neq 0$ inside R',

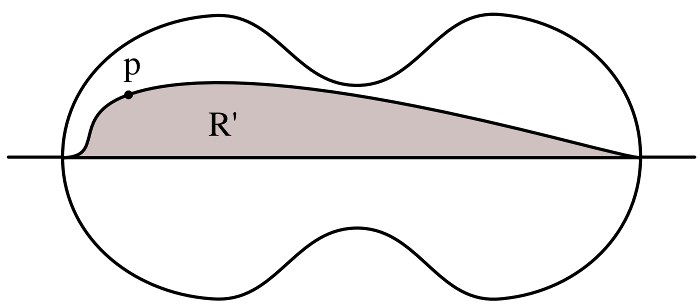


Figure 33: The second case where p is on an immersed arc.

and $L[f_z] = -\lambda^2 f_z$, it is clear that $-\lambda^2$ is the *first* eigenvalue of L on R', with Dirichlet boundary conditions. However, $-\lambda^2$ is also the first eigenvalue of L on the larger region

R. This contradicts the monotonicity principle for first eigenvalues of elliptic operators [**2**], which states that since R' is contained inside R, the first eigenvalue of L on R' must be strictly larger (in absolute value) than the first eigenvalue of L on R. We now pause to prove the various claims we made during this argument.

10. The function f_z is an eigenfunction of L

We first prove that f_z is an eigenfunction of L:

$$\begin{aligned} L[f_z] &= (f_z)_{rr} + (f_z)_{zz} - \frac{1}{r}(f_z)_r \\ &= (f_{rr})_z + (f_{zz})_z - \left[\frac{1}{r} f_r\right]_z \\ &= L[f]_z = -\lambda^2 f_z. \end{aligned}$$

11. No local max of f is an isolated zero of f_z

Suppose p is a local max for f, and an isolated zero of f_z. The point p must be a local max or local min for f_z. We restrict our attention to the line through p in the z-direction.

On the line, p is local max for f, so f_z changes sign at p. On the other hand, p is a local max or min for f_z, so f_z has the same sign in a neighborhood of p. The contradiction proves our claim.

12. Symmetry of f

We now show that f is symmetric over the r axis. Let $g(r,z) = f(r,-z)$. We have

$$g_r(r,z) = f_r(r,-z), \quad g_{rr}(r,z) = f_{rr}(r,-z), \quad g_{zz}(r,z) = f_{zz}(r,-z).$$

Thus $L[g] = -\lambda^2 g$. But $-\lambda^2$ is the first eigenvalue of L, and this eigenvalue is simple by Proposition 6 of Chapter 5. Thus, g must be a scalar multiple of f. Since $g = f$ on the r axis, $g = f$ everywhere. Thus

$$f(r,-z) = g(r,z) = f(r,z),$$

and f has a flip symmetry over the r axis. Further, f_z vanishes on the r axis.

13. Completing the argument

We have now shown that every local max of f lies on the r-axis. Suppose some saddle point p for f lies off the r-axis. Consider the singular level curve C of f through p. By Theorem 10, this level curve is an immersed circle. Let's consider the regions bounded by C which meet at p. Since each region contains a local max for f, each is cut by the r-axis. Further, C must have a flip symmetry over the r-axis. Thus, a portion of C appears as in Figure 13. Drawing the graph of this portion of C, we see a cycle, as in Figure 13, contradicting Theorem 11.

This completes the proof of Theorem 12.

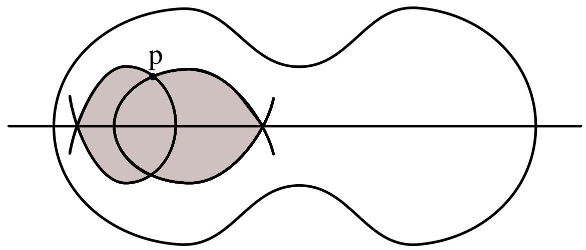


Figure 34: A portion of C.

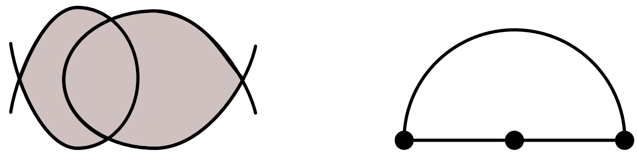


Figure 35: A portion of C and a portion of the graph $G(C)$.

CHAPTER 7

The Field on Each Integral Surface

1. Introduction

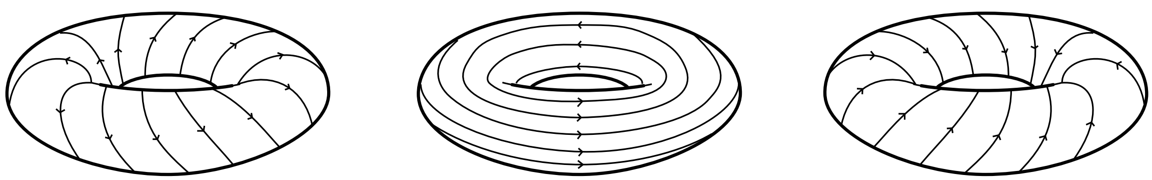
In the last chapter, we proved that every eigenfield

$$V = u(r,z)\hat{r} + v(r,z)\hat{\phi} + w(r,z)\hat{z},$$

of curl in $\mathrm{SAK}(\Omega)$ is tangent to the level surfaces of the function $rv(r,z)$. The fact that $rv(r,z)$ is a real-analytic function of r and z gave us some information about these level surfaces, and we combined that knowledge with a montonicity principle to prove that every integral surface of the first eigenfield was a smoothly immersed torus. Further, we showed that most integral surfaces of the first eigenfield are smoothly embedded tori, and that the singular integral surfaces are isolated. We also drew various other conclusions about the presence and structure of singular tori.

In this chapter, we want to investigate the behavior of the first eigenfield on an embedded integral torus T. In particular, we will show that the orbits of this field form right-handed helices on T. The key idea of the proof is a comparison argument: if the orbits of V formed *left*-handed helices on T, we would be able to construct another vector field V' on Ω with the same energy, but greater helicity.

The situation is summarized by the pictures below:



Possible orbits. Forbidden orbits.

Figure 36: Permitted and forbidden orbits on integral tori.

THEOREM 13. *Let V be the first eigenfield of curl in $\mathrm{SAK}(\Omega)$. Then on each nonsingular integral torus, the orbits of V are roughly helical, and these helices are always right-handed.*

Furthermore, on each integral torus, either all of the orbits of V are closed, or none are.

Recall that a torus T is said to be nonsingular if ∇rv never vanishes on T. We proved in the last Chapter that such a torus is always embedded. We call an orbit of V *roughly helical* if its tangent vector always has a positive component in the $\hat{\phi}$ direction, but never points solely in that direction

2. Background for the Proof of Theorem 13

Let T be a nonsingular integral torus of the first eigenfield V. Our goal is to show that the orbits of V form right-handed helices on T.

Since the norm of the component of V in the r-z plane is a positive multiple of $|\nabla rv|$ (by Proposition 4 of Chapter 4), and the component in the $\hat{\phi}$ direction is strictly positive (by Theorem 5 of Chapter 5), the orbits of V must form helices on T, and cannot form a family of circles.

Suppose that these helices are *left*-handed. By continuity, the orbits of V form left-handed helices on all the nonsingular tori in some neighborhood N of T.

We now divide Ω into three regions, E, N, and I, as below: Topologically, N and E

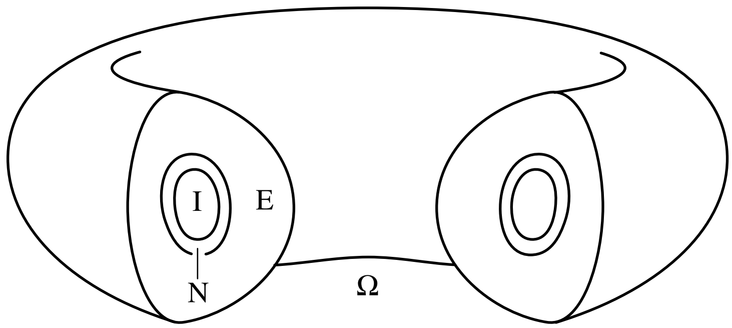


Figure 37: The regions E, N, and I.

are toroidal shells, while I is a solid torus.

3. Rewriting the Helicity of V

We now want to rewrite the helicity integral

$$\mathrm{H}(V) = \frac{1}{4\pi} \int_{\Omega\times\Omega} V(x) \times V(y) \cdot \frac{x-y}{|x-y|^3} \, d\,\mathrm{Vol}_x \; d\,\mathrm{Vol}_y \, .$$

Since we have divided Ω into E, N, and I, we can divide $\Omega \times \Omega$ into nine pieces:

$$\Omega \times \Omega = \begin{array}{ccc} E \times E & E \times N & E \times I \\ N \times E & N \times N & N \times I \\ I \times E & I \times N & I \times I \end{array}$$

and integrate separately over each one. Let V_E, V_N, and V_I be the portions of V supported in E, N, and I. Since the helicity integrand is symmetric in x and y, we can write

$$\mathrm{H}(V) = \begin{array}{ccc} \mathrm{H}(V_E) + 2\,\mathrm{H}(V_E, V_N) & + 2\,\mathrm{H}(V_E, V_I) \\ + \quad \mathrm{H}(V_N) & + 2\,\mathrm{H}(V_N, V_I) \\ & + \quad \mathrm{H}(V_I). \end{array}$$

Since V is tangent to the boundaries of E, N, and I, the Cross-Helicity Theorem of the Appendix tells us that the cross integrals can be written in terms of the fluxes of V over certain spanning surfaces in the three domains.

We make the choice of (oriented) spanning surfaces shown in Figure 3. and get the

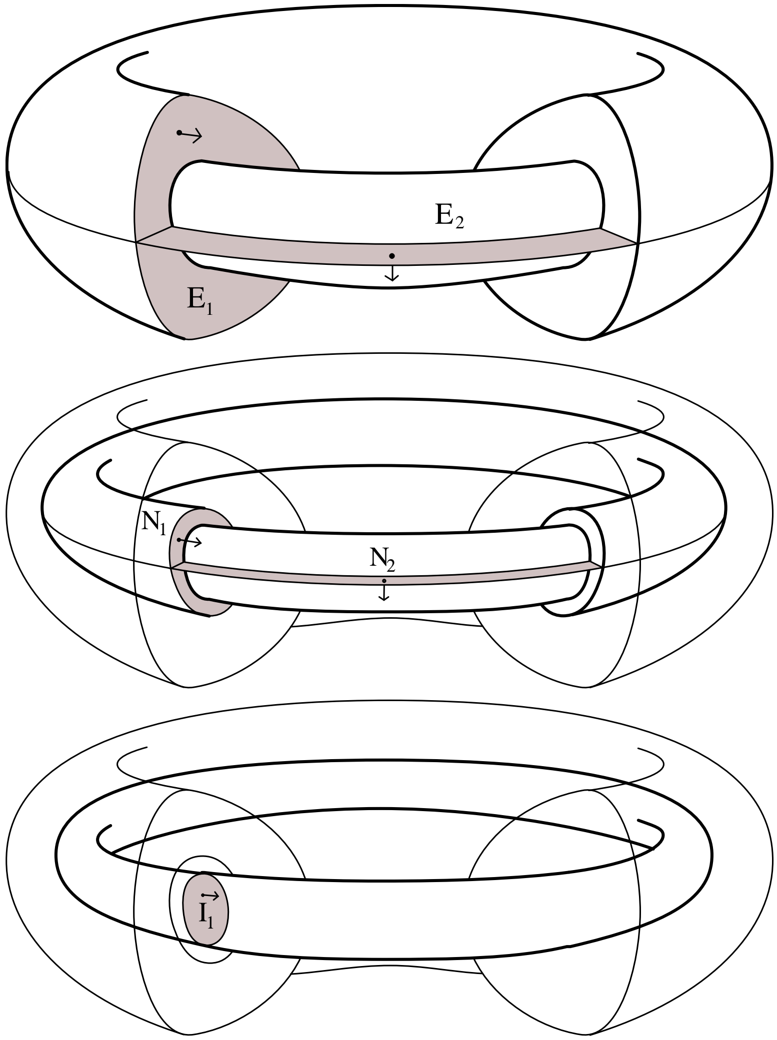


Figure 38: Spanning surfaces for E, N, and I.

following formulae:

$$\mathrm{H}(V_E, V_N) = \mathrm{Flux}(V_E, E_2)\,\mathrm{Flux}(V_N, N_1)$$
$$\mathrm{H}(V_E, V_I) = \mathrm{Flux}(V_E, E_2)\,\mathrm{Flux}(V_I, I_1)$$
$$\mathrm{H}(V_N, V_I) = \mathrm{Flux}(V_N, N_2)\,\mathrm{Flux}(V_I, I_1)$$

4. The competing field V'

We now define another vector field V' on Ω. Let V' be equal to V outside N. Inside N, if

$$V = u(r,z)\hat{r} + v(r,z)\hat{\phi} + w(r,z)\hat{z},$$

and

$$V' = u'(r,z)\hat{r} + v'(r,z)\hat{\phi} + w'(r,z)\hat{z},$$

then we set

$$u'(r,z) = -u(r,z), \quad v'(r,z) = v(r,z), \quad w'(r,z) = -w(r,z).$$

Clearly, V' is discontinuous at the boundary of N. However, V' is certainly L^2 integrable, and the energy of V' is equal to the energy of V.

We now show that $\mathrm{H}(V') > \mathrm{H}(V)$. Using the results of the last section, since $V' = V$ on E and I we have

$$\begin{aligned}\mathrm{H}(V') - \mathrm{H}(V) = &\, 2[\mathrm{H}(V_E, V'_N) - \mathrm{H}(V_E, V_N)] \\ &+ \mathrm{H}(V'_N) - \mathrm{H}(V_N) \\ &+ 2[\mathrm{H}(V'_N, V_I) - \mathrm{H}(V_N, V_I)].\end{aligned}$$

We deal with these terms in order. By our results above,

$$\mathrm{H}(V_E, V'_N) - \mathrm{H}(V_E, V_N) = \mathrm{Flux}(V_E, E_2)[\mathrm{Flux}(V'_N, N_1) - \mathrm{Flux}(V_N, N_1)].$$

Since $\hat{\phi}$ is the normal vector for the spanning surface N_1,

$$\mathrm{Flux}(V'_N, N_1) = \int_{N_1} v'(r,z)\, \mathrm{dArea},$$

and

$$\mathrm{Flux}(V_N, N_1) = \int_{N_1} v(r,z)\, \mathrm{dArea}\,.$$

Since $v'(r,z) = v(r,z)$, these fluxes are equal, and this term vanishes.

Next, let's consider the difference of

$$H(V'_N) = \frac{1}{4\pi} \int_{N \times N} V'(x) \times V'(y) \cdot \frac{x-y}{|x-y|^3}\, d\,\mathrm{Vol}_x \; d\,\mathrm{Vol}_y,$$

and

$$H(V_N) = \frac{1}{4\pi} \int_{N \times N} V(x) \times V(y) \cdot \frac{x-y}{|x-y|^3}\, d\,\mathrm{Vol}_x \; d\,\mathrm{Vol}_y\,.$$

Both are integrals over $N \times N$, so they are proportional to the square of the volume of N. The other terms in our story, however, are integrals over $N \times I$ and $N \times E$, and are proportional to the first power of the volume of N. Since the volume of N can be chosen arbitrarily small, these integrals are insignificant when compared to the remaining terms in our story, and can be safely ignored.

For the last term in our formula, we examine

$$\mathrm{H}(V'_N, V_I) - \mathrm{H}(V_N, V_I) = \mathrm{Flux}(V_I, I_1)[\mathrm{Flux}(V'_N, N_2) - \mathrm{Flux}(V_N, N_2)].$$

Since the orbits of V are *left*-handed helices on N, the flux of V_N over N_2 must be strictly negative. Since the normal to N_2 is perpendicular to $\hat{\phi}$, only the components of V and V' in the $\hat{r}$ and $\hat{z}$ direction contribute to the flux integral, and $\mathrm{Flux}(V'_N, N_2) = -\mathrm{Flux}(V_N, N_2)$. Since $\hat{\phi}$ is the normal vector to I_1, and the component of V in that direction is always positive, $\mathrm{Flux}(V_I, I_1)$ is strictly positive. Thus, this term is strictly positive.

We have proven that $\mathrm{H}(V') > \mathrm{H}(V)$, and that $\mathrm{E}(V') = \mathrm{E}(V)$. We now argue that V' is in the L^2 closure of $\mathrm{SK}(\Omega)$.

Each component of ∂N is a level surface of the function rv. Suppose that $rv = a$ on the inner component, and $rv = b$ on the outer component.

Approximate V' by vector fields in the form $g(rv)V'$, where g is a smooth function, equal to one outside a small neighborhood of a and b, and vanishing in a smaller neighborhood of these values. It is clear that gV' is smooth, symmetric and tangent to $\partial\Omega$. Let's compute $\nabla \cdot gV'$.

Where g is zero, gV' is certainly divergence-free. Where g does not vanish, V' is smooth, and

$$\nabla \cdot gV' = g(\nabla \cdot V') + \nabla g \cdot V' = \nabla g \cdot V'.$$

But $\nabla g(rv) = g'(\nabla rv)$, and since ∇rv is in the r-z plane,

$$V' \cdot \nabla rv = -V \cdot \nabla rv = 0,$$

(using Theorem 9 of Chapter 6). Thus gV' is in $\mathrm{SK}(\Omega)$.

Choosing a family of functions f appropriately, we can easily generate a sequence of vector fields in $\mathrm{SK}(\Omega)$ converging to V' in the L^2 norm.

Theorem 4 of Chapter 3 tells us that V is the vector field in the L^2 closure of $\mathrm{SK}(\Omega)$ with least energy for its helicity. Since V' is in this closure, with the same energy as V and greater helicity, the existence of V' provides a contradiction. This proves that the orbits of V must form ***right***-handed helices on T.

5. All orbits of V are closed, or none are.

We now prove the second part of Theorem 13: every orbit of V on T is closed, or none are. Our first move is to replace V by the rescaled field $W = r^2V$. Since V and W point in the same direction, their orbits are the same.

Every particle moving in the flow W has angular velocity in the $\hat{\phi}$ direction given by $(1/r)W \cdot \hat{\phi}$. Since $W = r^2V$, this angular velocity is given by $r(V \cdot \hat{\phi})$. Of course, T is a level surface of the function $r(V \cdot \hat{\phi}) = rv$.

Thus, every particle moving in the flow W on T has constant angular velocity in the $\hat{\phi}$ direction. In particular, each particle makes a complete circle around the z axis in the same time t.

Choose a cross-sectional circle $C(T)$, and parametrize it by arc-length with parameter $s \in [0, S]$. The component of W in the r-z plane, denoted W_{rz}, is tangent to $C(T)$, and (since T is nonsingular) never vanishes on $C(T)$ (by Proposition 4 of Chapter 4).

Suppose that a particle p starts at position $s = a$ on $C(T)$ at time zero, and flows under W_{rz}. The particle will reach $s = a$ again at time

$$t_a = \int_a^S \frac{1}{|W_{rz}|}\,\mathrm{ds} + \int_0^a \frac{1}{|W_{rz}|}\,\mathrm{ds}\,.$$

Clearly, this time is finite, and does not depend on a.

Since W is rotationally symmetric, if p flows under W it will reach position $s = a$ on some cross section of T at time t_a. It is clear that p will eventually reach $s = a$ on the *original* cross-section (and close its orbit) if and only if t/t_a is rational.

Since this number is the same for all $p \in T$, either every orbit of W is closed, or none are.

CHAPTER 8

The Generalized Cross-Helicity Formula

1. Introduction

Suppose we have disjoint domains Ω and Ω' in $\mathbf{R}^3$ and divergence–free vector fields V and V' defined on Ω and Ω' and tangent to their boundaries. The cross-helicity of these fields is the integral

$$\mathrm{H}(V, V') = \frac{1}{4\pi} \int_{\Omega \times \Omega'} V(p) \times V'(q) \cdot \frac{p-q}{|p-q|^3} \, \mathrm{dVol}_p \, \mathrm{dVol}_q \, .$$

In 1992, with the assumption that Ω and Ω' were solid tori of "vanishingly small" cross-section, Ricca and Moffatt [**16**] proved that

$$\mathrm{H}(V, V') = \mathrm{Lk}(\Omega, \Omega') \, \mathrm{Flux}(V) \, \mathrm{Flux}(V').$$

This is an interesting and beautiful result, but the requirements on Ω and Ω' are very strict—we have prescribed both topology and geometry.

Our goal is to show that neither constraint is required.

We will be able to give a general formula for the cross-helicity $H(V, V')$ in terms of the linking numbers of curves constructed from Ω and Ω' and the fluxes of V and V' over corresponding spanning surfaces in those domains. This formula will apply to domains of arbitrary topology and geometry in 3-space.

2. Choosing Curves and Spanning Surfaces

We begin by choosing a basis for the first homology of Ω, represented by curves $C_1, \ldots, C_n$ on the boundary of Ω. These curves are dual to a set of spanning surfaces for Ω, denoted $S_1, \ldots, S_n$. Poincaré duality guarantees that these surfaces represent a basis for the second homology of Ω, relative to the boundary of Ω.

In practice, we choose the S_i by checking that the intersection number of C_i with S_j is δ_{ij}. We can double-check our work by verifying that n is equal to the total genus of the boundary of Ω.

3. Statement of the theorem

We can now state the theorem.

THEOREM 14. *If we choose C_i and S_i for Ω as above, and choose C'_j and S'_j for Ω' as well, the cross-helicity of a pair of divergence-free vector fields V and V', tangent to the smooth boundaries of two compact domains Ω and Ω' in 3-space, is given by*

$$\mathrm{H}(V, V') = \sum_{i,j} \mathrm{Lk}(C_i, C'_j) \, \mathrm{Flux}(V, S_i) \, \mathrm{Flux}(V', S'_j).$$

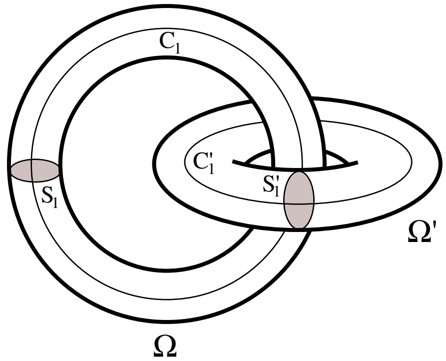


Figure 39: The case of Ricca and Moffatt.

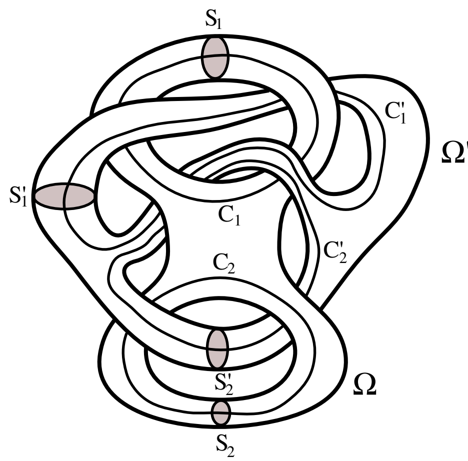


Figure 40: A more complicated domain.

Figure 3 illustrates the method in the case of Ricca and Moffatt.
In this case, $\mathrm{Lk}(C_1, C_1') = 1$, and so

$$\mathrm{H}(V, V') = \mathrm{Flux}(V, S_1)\,\mathrm{Flux}(V', S_1').$$

Now, to see the power of this method, we'll try a more complicated domain, shown in Figure 3, and get a more complicated formula.
Here, we have

$$\mathrm{Lk}(C_1, C_1') = 2.$$
$$\mathrm{Lk}(C_1, C_2') = 1.$$
$$\mathrm{Lk}(C_2, C_1') = 0.$$
$$\mathrm{Lk}(C_2, C_2') = -1,$$

so the helicity is given by

$$\begin{aligned}\mathrm{H}(V,V') &= 2\,\mathrm{Flux}(V,S_1)\,\mathrm{Flux}(V',S_1') \\ &\quad + \mathrm{Flux}(V,S_1)\,\mathrm{Flux}(V',S_2') \\ &\quad - \mathrm{Flux}(V,S_2)\,\mathrm{Flux}(V',S_2').\end{aligned}$$

We will start by recalling some standard results about the topology of domains in 3-space, and by recalling the standard correspondance between vector fields and cohomology.

Then we will be able to prove the theorem.

4. The topology, homology, and cohomology of domains in 3-space.

Take a compact domain Ω in 3-space, with boundary $\partial\Omega$, such as a ball, or a solid torus. The boundary is a surface, or a collection of surfaces, and each surface is topologically equivalent to a torus with a number of holes. The number of holes is called the *genus* of the surface. For example, each domain in the example above is bounded by a surface of genus two.

We need to study the homology and cohomology of these domains, with real coefficients. In this world, the first homology of Ω is a real vector space of equivalence classes of closed, oriented curves in Ω, where two curves are considered equivalent if they bound a common surface inside the domain, and the sign of a curve depends on its orientation. The first homology of Ω is denoted $H_1(\Omega)$.

The second homology of Ω, relative to the boundary of Ω, is a vector space of equivalence classes of "spanning surfaces" for Ω, whose boundaries lie on the boundary of Ω. Two such surfaces are considered equivalent if they, together with any portion of $\partial\Omega$, bound a common volume inside Ω. Again, the sign of a surface depends on its orientation. The second relative homology of Ω is denoted $H_2(\Omega, \partial\Omega)$.

The cohomology spaces of Ω are the dual vector spaces to the homology vector spaces. A cohomology class is a linear functional on homology classes of the right dimension and kind. Our cohomology spaces of Ω are denoted $H^1(\Omega)$ and $H^2(\Omega, \partial\Omega)$. There is an operation, called the cup product, which takes a class in $H^1(\Omega)$, together with a class in $H^2(\Omega, \partial\Omega)$, and (effectively) returns a real number.

If a curve C is dual to $C^* \in H^1(\Omega)$, and a spanning surface S is dual to $S^* \in H^2(\Omega, \partial\Omega)$, then the cup product $C^* \cup S^*$ is equal to the intersection number of C and S. This intersection number is just the sum of the times that C punctures S, where a puncture adds one to the sum if the unit tangent vector to C agrees with the unit normal to S at the puncture, and subtracts one if these vectors disagree. This intersection number is equal to the linking number of C with the boundary of S.

A beautiful result called Poincaré duality tells us that the vector spaces $H_1(\Omega)$ and $H_2(\Omega, \partial\Omega)$ are isomorphic. Further, if we take a basis for $H_1(\Omega)$ represented by curves $C_1, \ldots, C_n$, and the corresponding dual basis for $H_2(\Omega, \partial\Omega)$, represented by surfaces $S_1, \ldots, S_n$, the intersection number of C_i and S_j will be δ_{ij}.

5. Vector fields and cohomology

There is also a canonical correspondence between vector fields and cohomology classes. A vector field V on Ω defines a linear functional on closed curves inside Ω by taking the circulation of V around the curve:

$$V(C) = \int_C V \cdot \mathrm{ds}\,.$$

If V is curl-free on Ω, then V defines a linear functional on homology classes of curves, for if C and C' together bound S, then by Stokes' Theorem we have

$$V(C) - V(C') = \int_C V \cdot \mathrm{ds} - \int_{C'} V \cdot \mathrm{ds} = \int_S \nabla \times V \cdot n \, \mathrm{dArea} = 0.$$

Thus a curl-free vector field corresponds to a class in $H^1(\Omega)$.

A vector field V on Ω defines a linear functional on spanning surfaces inside Ω by taking the flux of V across the surface:

$$V(S) = \int_S V \cdot n \, \mathrm{dArea}\,.$$

If V is divergence-free and tangent to the boundary of Ω, then V defines a linear functional on homology classes of spanning surfaces, for if S and S', together with some portion B of $\partial\Omega$, bound a volume D inside Ω, then by the Divergence Theorem, we have

$$\int_S V \cdot n \, \mathrm{dArea} - \int_{S'} V \cdot n \, \mathrm{dArea} + \int_B V \cdot n \, \mathrm{dArea} = \int_D \nabla \cdot V = 0.$$

But V is tangent to $\partial\Omega$, so this proves that

$$V(S) - V(S') = \int_S V \cdot n \, \mathrm{dArea} - \int_{S'} V \cdot n \, \mathrm{dArea} = 0.$$

Thus, a divergence-free vector field, tangent to $\partial\Omega$, corresponds to a class in $H^2(\Omega, \partial\Omega)$.

In the world of vector fields, the cup product corresponds to the L^2 inner product:

$$V \cup W = \langle V, W \rangle = \int_\Omega V \cdot W \, \mathrm{dVol}\,.$$

We have now introduced all the technology we'll need to prove our formula.

6. Rewriting the Helicity Integral

It's standard to rewrite the helicity integral $\mathrm{H}(V, V')$ as:

$$\begin{aligned} \mathrm{H}(V, V') &= \frac{1}{4\pi} \int_{\Omega \times \Omega'} V(p) \times V'(q) \cdot \frac{p - q}{|p - q|^3} \, \mathrm{dVol}_p \, \mathrm{dVol}_q \\ &= \int_\Omega V(p) \cdot \left[\frac{1}{4\pi} \int_{\Omega'} V'(q) \times \frac{p - q}{|p - q|^3} \, \mathrm{dVol}_q \right] \mathrm{dVol}_p \\ &= \langle V, \mathrm{BS}(V') \rangle \,, \end{aligned}$$

where, if we view V' as an electrical current on Ω', $\mathrm{BS}(V')$ is the resulting magnetic field in 3-space. It's a well known fact from physics that the curl of the magnetic field $\mathrm{BS}(V')$ is equal to the current flow V' on Ω, and that the curl $\mathrm{BS}(V')$ vanishes elsewhere. In addition, the the magnetic field $\mathrm{BS}(V')$ is everywhere divergence-free.

7. Reduction to a product of harmonic knots

We now set out to prove our result. We start with the Hodge Decomposition Theorem [**7**].

THEOREM 15. *We have a direct sum decomposition of* $\mathrm{VF}(\Omega)$ *into five mutually orthogonal subspaces,*

$$\mathrm{VF}(\Omega) = \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG},$$

with

$$\begin{aligned} \mathrm{Ker\,Curl} &= \phantom{\mathrm{FK} \oplus{}} \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG} \\ \mathrm{Image\,Grad} &= \phantom{\mathrm{FK} \oplus \mathrm{HK} \oplus{}} \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG} \\ \mathrm{Image\,Curl} &= \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG} \\ \mathrm{Ker\,Div} &= \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG} \end{aligned}$$

where

$$\begin{aligned} \mathrm{FK} &= \{\nabla \cdot V = 0,\ V \cdot n = 0,\ \text{all interior fluxes } = 0\}, \\ \mathrm{HK} &= \{\nabla \cdot V = 0, \nabla \times V = 0, V \cdot n = 0\}, \\ \mathrm{CG} &= \{V = \nabla\varphi, \nabla \cdot V = 0, \text{all boundary fluxes} = 0\}, \\ \mathrm{HG} &= \{V = \nabla\varphi, \nabla \cdot V = 0, \varphi \text{ loc. constant on } \partial\Omega\}, \\ \mathrm{GG} &= \{V = \nabla\varphi,\ \varphi|_{\partial\Omega} = 0\}, \end{aligned}$$

and furthermore,

$$\begin{aligned} \mathrm{HK} &\cong H_1(\Omega; \mathbf{R}) \cong H_2(\Omega, \partial\Omega; \mathbf{R}) \\ &\cong \mathbf{R}^{\text{genus of } \partial\Omega}. \\ \mathrm{HG} &\cong H_2(\Omega; \mathbf{R}) \cong H_1(\Omega, \partial\Omega; \mathbf{R}) \\ &\cong \mathbf{R}^{(\#\text{ components of } \partial\Omega) - (\#\text{ components of } \Omega)}. \end{aligned}$$

We refer to FK as "fluxless knots", HK as "harmonic knots", CG as "curly gradients", HG as "harmonic gradients", and GG as "grounded gradients".

Since V is divergence-free and tangent to $\partial\Omega$, $V \in \mathrm{FK} \oplus \mathrm{HK}$. Since $\mathrm{BS}(V')$ is divergence and curl free on Ω, $\mathrm{BS}(V') \in \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG}$. This means that the inner product of V and $\mathrm{BS}(V')$ is completely determined by the inner product of the components of these fields in the subspace HK.

8. Projection of V to HK

We just observed that $V = F + H$, where $F \in \mathrm{FK}$ and $H \in \mathrm{HK}$. Let's write down a lemma.

LEMMA 4. *Given any* S_i *in our basis for* $H_2(\Omega, \partial\Omega)$*, for* V *and* H *as above, we have*

$$\mathrm{Flux}(V, S_i) = \mathrm{Flux}(H, S_i).$$

Since $V = F + H$, and F is fluxless by construction, this is easy.

9. Projection of $BS(V')$ to HK

We also know that $\mathrm{BS}(V') = H + \nabla f$, where $H \in \mathrm{HK}$. We have a corresponding lemma.

LEMMA 5. *Given a closed curve C_i in our basis for $H_1(\Omega)$, if*

$$\mathrm{BS}(V') = H + \nabla f,$$

where $H \in \mathrm{HK}$, then

$$\mathrm{Circ}(\mathrm{BS}(V'), C_i) = \mathrm{Circ}(H, C_i).$$

where $\mathrm{Circ}(V, C)$ denotes the circulation of the vector field V around the closed curve C.

Since ∇f is a gradient, $\mathrm{Circ}(\nabla f, C_i) = 0$.

10. Inner Products of Harmonic Knots

We now find a general formula for the inner product of two vector fields in $\mathrm{HK}(\Omega)$. To do so, we'll rederive Proposition 5, from [**7**], which was inspired by a corresponding result of Blank, Friedrichs, and Grad [**3**].

PROPOSITION 11. *Suppose C_i and S_i represent bases for $H_1(\Omega)$ and $H_2(\Omega, \partial\Omega)$, chosen as above. Then, for any pair of harmonic knots V and W, we have*

$$\langle V, W \rangle = \sum_i \mathrm{Circ}(V, C_i)\,\mathrm{Flux}(W, S_i).$$

When we introduced the correspondence between vector fields and cohomology classes, we mentioned that curl-free vector fields were equivalent to one-dimensional cohomology classes, and divergence-free vector fields tangent to $\partial\Omega$ were equivalent to two-dimensional relative cohomology classes. Members of HK are *both* curl-free *and* divergence-free and tangent to $\partial\Omega$. Thus, we are free to view V as a one-dimensional class, and W as a two-dimensional relative class.

Now the C_i and S_i represent bases for $H_1(\Omega)$ and $H_2(\Omega, \partial\Omega)$, so their duals C_i^* and S_i^* represent bases for $H^1(\Omega)$ and $H^2(\Omega, \partial\Omega)$. Writing $V = \sum c_i C_i^*$ and $W = \sum s_i S_i^*$ in these dual bases, it is clear that

$$c_i = \mathrm{Circ}(V, C_i), \text{ and } s_i = \mathrm{Flux}(W, S_i).$$

The inner product $\langle V, W \rangle$ is equal to the cup product $V \cup W$. But the cup product $C_i^* \cup S_j^*$ is just the intersection number of C_i and S_j, and since we've chosen the S_j to be the Poincaré duals of the C_i, this intersection number is δ_{ij}. Thus

$$\begin{aligned}
\langle V, W \rangle = V \cup W &= \left(\sum_i c_i C_i^*\right) \cup \left(\sum_j s_j S_j^*\right) \\
&= \sum_{ij} c_i s_j \left(C_i^* \cup S_j^*\right) = \sum_{ij} c_i s_j \delta_{ij} \\
&= \sum_i c_i s_i = \sum_i \mathrm{Circ}(V, C_i)\,\mathrm{Flux}(W, S_i).
\end{aligned}$$

11. Combining results obtained so far

Combining Lemma 4 and Lemma 5 with Proposition 11, we have proved that

$$\langle V, \mathrm{BS}(V') \rangle = \sum_i \mathrm{Flux}(V, S_i)\, \mathrm{Circ}(\mathrm{BS}(V'), C_i).$$

Our goal now must be to prove that

$$\mathrm{Circ}(\mathrm{BS}(V'), C_i) = \sum_j \mathrm{Lk}(C_i, C'_j)\, \mathrm{Flux}(V', S'_j).$$

We will show in Proposition 12 that every homology class of curves in Ω is represented by the boundary of some spanning surface E outside Ω. Let C_i reside in the same homology class as ∂E_i. Then by Stokes' theorem,

$$\mathrm{Circ}(\mathrm{BS}(V'), C_i) = \mathrm{Circ}(\mathrm{BS}(V'), \partial E_i) = \mathrm{Flux}(\nabla \times \mathrm{BS}(V'), E_i).$$

But $\nabla \times \mathrm{BS}(V')$ is equal to V' inside Ω', and is zero elsewhere. We've proved that

$$\mathrm{Circ}(\mathrm{BS}(V'), C_i) = \mathrm{Flux}(V', E_i).$$

It remains to compute the fluxes of V' over the E_i.

12. The flux of V' over E_i

Suppose that we've chosen curves C'_i and Poincaré dual surfaces S'_i representing bases for the homology of Ω'. The intersection of E_i with Ω' is certainly a spanning surface inside Ω', so it must be equivalent to a linear combination of the surfaces S'_i. That is,

$$E_i = \sum_j e_{ij} S'_j.$$

To compute the coefficients e_{ij}, compute the intersection number of C'_j with E_i; since the S'_j are Poincaré duals of the C'_j, this intersection number is equal to e_{ij}. Since C_i is the boundary of E_i, this proves that

$$E_i = \sum_j \mathrm{Int}(C'_j, E_i) S'_j = \sum_j \mathrm{Lk}(C'_j, \partial E_i) S'_j = \sum_j \mathrm{Lk}(C'_j, C_i) S'_j.$$

In particular, the flux of V' over E_i is given by

$$\mathrm{Flux}(V', E_i) = \sum_j \mathrm{Lk}(C_i, C'_j)\, \mathrm{Flux}(V', S'_j),$$

which completes the proof of the Theorem.

13. The Connecting Homomorphism

In Section 11, we claimed that every homology class of curves in $H_1(\Omega)$ was represented by the boundary of some spanning surface in $\mathbf{R}^3 - \Omega$. We now prove this result, drawing on some of the standard tools of algebraic topology:

PROPOSITION 12. *Consider the boundary map δ, which replaces a spanning surface in* $\mathbf{R}^3 - \Omega$ *with it's boundary curve on* $\partial\Omega$. *The induced map* $H_2(\mathbf{R}^3 - \Omega, \partial\Omega) \to H_1(\partial\Omega)$, *when composed with the map induced by inclusion,* $H_1(\partial\Omega) \to H_1(\Omega)$, *yields an isomorphism*

$$H_2(\mathbf{R}^3 - \Omega, \partial\Omega) \cong H_1(\Omega).$$

Our first tool is the Mayer-Vietoris exact sequence for homology. In our case, this sequence becomes

$$\to H_2(\mathbf{R}^3) \to H_1(\partial\Omega) \to H_1(\Omega) \oplus H_1(\mathbf{R}^3 - \Omega) \to H_1(\mathbf{R}^3) \to$$

Since $H_2(\mathbf{R}^3) = H_1(\mathbf{R}^3) = 0$, the middle arrow must be an isomorphism between $H_1(\partial\Omega)$ and $H_1(\Omega) \oplus H_1(\mathbf{R}^3 - \Omega)$. This isomorphism is induced by the inclusion maps from $\partial\Omega$ into Ω and into $\mathbf{R}^3 - \Omega$.

We can use Mayer-Vietoris to derive a similar isomorphism between $H_2(\partial\Omega)$ and $H_2(\Omega) \oplus H_2(\mathbf{R}^3 - \Omega)$, again induced by the inclusions.

Our next tool is the relative homology exact sequence for the pair ($\mathbf{R}^3 - \Omega$, $\partial\Omega$). The relevant portion reads

$$H_2(\partial\Omega) \xrightarrow{a} H_2(\mathbf{R}^3 - \Omega) \xrightarrow{b} H_2(\mathbf{R}^3 - \Omega, \partial\Omega) \xrightarrow{\delta} H_1(\partial\Omega) \xrightarrow{c} H_1(\mathbf{R}^3 - \Omega)$$

We first prove that the arrow marked δ is one-to-one. It suffices to show that b is the zero map. But the map a comes from the inclusion $\partial\Omega \hookrightarrow \mathbf{R}^3 - \Omega$, so a must be onto by the second Mayer-Vietoris argument above.

We have proved that δ is an isomorphism between $H_2(\mathbf{R}^3 - \Omega, \partial\Omega)$ and the image of δ inside $H_1(\partial\Omega)$. Yet the image of δ is the kernel of the arrow marked c above.

And by the first Mayer-Vietoris argument above, since c is induced by the inclusion $\partial\Omega \hookrightarrow \mathbf{R}^3 - \Omega$, the kernel of c is $H_1(\Omega) \subset H_1(\partial\Omega)$.

This completes the proof of the proposition.

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