Parametrized curves, examples and constructions

Recall that a parametrized curve is a map \( \vec{\alpha} : \mathbb{R} \to \mathbb{R}^2 \). We will now study some example curves.

Example. The circle of radius \( r \) with center \( \vec{c} = (c_1, c_2) \) in \( \mathbb{R}^2 \) is described implicitly by

\[
(x - c_1)^2 + (y - c_2)^2 = r^2
\]

We can parametrize this curve by

\[
\vec{\alpha} + r (\cos t, \sin t) = \vec{\alpha}(t)
\]
Notice that
\[ \vec{\alpha}(t) = (c_1 + r\cos t, c_2 + rs\sin t) \]
obeys
\[
(\alpha_1(t) - c_1)^2 + (\alpha_2(t) - c_2)^2 = \\
= (r\cos t)^2 + (rs\sin t)^2 \\
= r^2 (\cos^2 t + \sin^2 t) = r^2,
\]
but there is more information in the parametrization \( \vec{\alpha}(t) \) because it tells us when each point on the circle is reached.

Example 2. \( \vec{\alpha}(t) = (c_1 + r\cos(t^2), c_2 + rs\sin(t^2)) \) also parametrizes the circle of radius \( r \) and center \( \hat{c} = (c_1, c_2) \).
We can make some beautiful curves by combining sines and cosines.

Example. A unit circle starts with center at \((0,1)\) and rolls along the positive \(x\) axis. Parametrize the path of a point starting at \((1,1)\).

If the center of the circle is given by \(\hat{\mathbf{r}}(t)\), we can assume that the circle is rolling to the right at unit speed, so \(\hat{\mathbf{r}}(t) = (t, 1)\).
However, if a unit circle has rolled $t$ units forward, it has turned by an angle of $t$ radians... in the **clockwise** direction.

This rotation carries the point at $(1,0)$ to (relative to the center) to the point at $(\cos \theta, -\sin \theta)$ (relative to the center).

Adding these together:

$$\vec{x}(t) = (t + \cos \theta, 1 - \sin t)$$
We will work through a more elaborate example of this type of motion in homework when we describe the square-wheeled car.

We often describe curves with a differential equation, so let's remember how to solve (easy) ODEs.

If \( u'(t) = F(u(t)) \), then

\[
\frac{u'(t)}{F(u(t))} = \frac{1}{F(u(t))}
\]

Integrate w.r.t. \( t \)

\[
\int \frac{u'(t)}{F(u(t))} \, dt = \int \frac{1}{F(u(t))} \, dt
\]

\[
\int \frac{1}{F(u)} \, du = \int 1 \, dt
\]

So

\[
\int \frac{1}{F(u)} \, du = t + C
\]
If we can do the integral on the left to get some
\[ G(u) = \int \frac{1}{F(u)} \, du \]
then we get an equation
\[ G(u) = t \]
which we can try to solve for \( u(t) \).

Example. \( u'(t) = u(t)^2 \)

\[ \frac{u'(t)}{u(t)^2} = 1 \quad \implies \quad \int \frac{1}{u(t)^2} u'(t) \, dt = \int 1 \, dt \]

\[ \implies \int \frac{1}{u^2} \, du = t + C \quad \implies \quad -\frac{1}{u} = t + C \]

\[ \implies u = -\frac{1}{t + C} \]

So \( u(t) = -\frac{1}{t + C} \), and indeed \( u'(t) = \frac{1}{(t + C)^2} = u(t)^2 \).