Math 4250: The dot product, the point groups, and the regular solids.

**Definition 1** \( P \subset \mathbb{R}^3 \) is a regular solid if every face is an identical regular polygon and the same number of faces meet at each vertex.

You probably remember that there are only 5 regular solids: the tetrahedron, cube, octahedron, dodecahedron, and icosahedron, and I suspect that you can picture them. But here’s a harder question: what are the coordinates of their vertices? Surely they should have some beautiful structure of their own! One way to generate coordinates is to use the point groups. For instance, any four (noncoplanar) points \( \vec{p}, \vec{q}, \vec{r}, \vec{s} \in \mathbb{R}^3 \) form a tetrahedron by taking the four triangular faces to be \( \{\vec{q}, \vec{r}, \vec{s}\} \), \( \{\vec{p}, \vec{r}, \vec{s}\} \), \( \{\vec{p}, \vec{q}, \vec{s}\} \), and \( \{\vec{p}, \vec{q}, \vec{r}\} \), as below.

![Diagram of a tetrahedron]

Since three triangles meet at each vertex, the second condition of Definition 1 is met regardless of the positions of \( \vec{p}, \vec{q}, \vec{r}, \vec{s} \). However, the triangular faces may not all be equilateral.

![Diagram of vertices and unit vectors]

Here, \( \vec{p} = (1, 1, 1) \), \( \vec{q} = (-1, -1, 1) \), \( \vec{r} = (1, -1, -1) \) and \( \vec{s} = (-1, 1, -1) \). This is a special tetrahedron!
1. (10 points) We are now going to use the point group $G$ to show that the tetrahedron above with $\vec{p} = (1, 1, 1)$, $\vec{q} = (-1, -1, 1)$, $\vec{r} = (1, -1, -1)$ and $\vec{s} = (-1, 1, -1)$ is a regular solid.

(a) (5 points) Prove that $A$ and $B$ permute $\vec{p}$, $\vec{q}$, $\vec{r}$, and $\vec{s}$ and write down the permutation. Conclude that every element of $G$ permutes these vectors in some way (because it is a product of $A$’s and $B$’s).
(b) (5 points) Use the last question to find an isometry in $G$ (that is, a product of $A$’s and $B$’s) which takes the edge $\{\vec{p}\vec{q}\}$ to each of the other 5 edges of the tetrahedron: $\{\vec{p},\vec{r}\}$, $\{\vec{p}\vec{s}\}$, $\{\vec{q}\vec{r}\}$, $\{\vec{r}\vec{s}\}$, and $\{\vec{s}\vec{r}\}$ to show that all the faces are equilateral triangles.
2. (10 points) Starting with any \( \vec{v} = (1, x, 0) \) (assume \( x < 1 \)), we can generate 12 vectors \( \vec{v}_1, \ldots, \vec{v}_{12} \) by applying the 12 matrices in \( \mathcal{G} \) to \( \vec{v} \). We can group these into the vertices of 3 rectangles in the \( \vec{e}_1 - \vec{e}_2, \vec{e}_2 - \vec{e}_3 \) and \( \vec{e}_3 - \vec{e}_1 \) planes as below.

Label each vertex above with its coordinates and the corresponding matrix in \( \mathcal{G} \) (written as a product of \( A \)'s and \( B \)'s). We have labeled \( I\vec{v} \) and \( A\vec{v} \) above to help you get started.
3. (10 points) As we did with the tetrahedron, we’re now going to use the point group to show that certain distances between our 12 points are the same and we’re going to connect this group to a different Platonic solid!

(a) (5 points) The edge \( \vec{A} \vec{B} \) is one of 12 edges marked in blue on the picture below. Use the results of Question 2 to describe each of these edges in the form \( \{C \vec{v} D \vec{v}\} \) where \( C \) and \( D \) are products of \( A \)'s and \( B \)'s.
(b) (5 points) Prove that all of these edges have the same length by finding isometries in $G$ which take $\{I\vec{v}, A\vec{v}\}$ to each of these edges. This proves that the blue triangles are equilateral. Hint: You’ll eventually need to use the relations between products of $A$ and $B$ that you developed from $(AB)^3 = I$ in the course of Question ??.
(c) (5 points) You don’t have to match isometries with edges explicitly again, but the 12 isometries in $\mathcal{G}$ map the edge \{\vec{v}, BAB\vec{v}\} to the 12 edges in red below left. The red edges all have the same length and the red triangles are equilateral. So our construction yields a one-parameter family of solids which are $\mathcal{G}$-symmetric, depending on the $x$ in $\vec{v} = (1, x, 0)$.

Each has 12 vertices and 20 triangular faces, with 5 triangles meeting at every vertex. However, while the 4 red triangles and the 4 blue triangles are always equilateral, the 12 green triangles are only isosceles. An example is shown below center.

Solve for the length of the red edges $r(x)$ and the blue edges $b(x)$ to prove that $b(x) = r(x)$. Then set $b(x) = r(x) = 2x$ (the length of the short side of the rectangles) to find the $x$ which makes the green triangles equilateral and the entire figure an icosahedron, as shown below right.

Hint: The value $x$ should be familiar ... what is it?