Math 4250/6250: Getting comfortable again with linear algebra

We are going to warm up with a few easy questions about the relationship between dot products and matrix products.

Proposition 1 *Here are some standard facts from linear algebra:*

1. *We identify vectors* $\vec{v} = (v_1, \dots, v_n)$ *with* $n \times 1$ *matrices* $V = (v_1 \dots v_n)^T$.
2. *If* A *is an* $n \times m$ *matrix and* B *is an* $m \times p$ *matrix, then their matrix product* AB *is the* $n \times p$ *matrix defined by*
$$(AB)_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}.$$
3. *The dot product is defined by* $\langle \vec{v}, \vec{w} \rangle = \sum_i v_i w_i = (v_1 \dots v_n)(w_1 \dots w_n)^T = V^T W$.
4. *Matrix-vector multiplication is defined by*
$$A\vec{v} = AV = \left(\sum_k a_{1k} v_k \ \dots \ \sum_k a_{nk} v_k \right)^T.$$
5. *We let* $\vec{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ *where the* 1 *is in the* i*-th position. The vectors* $\vec{e}_1, \dots, \vec{e}_n$ *form the standard basis for* $\mathbb{R}^n$. *As matrices, they are given by* $E_i = (\delta_{i1} \dots \delta_{in})^T$ *where* δ_{ik} *is the Kronecker delta:*
$$\delta_{ik} = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{otherwise.} \end{cases}$$

1. (10 points) We begin with working out some basic properties from linear algebra:

   (1) (5 points) Suppose that A is an $n \times n$ matrix. Use our linear algebra facts to show

$$A_{ij} = \langle \vec{e}_i, A\vec{e}_j \rangle \, .$$

(2) (5 points) Suppose that A is an $n \times n$ matrix and $\vec{v}, \vec{w} \in \mathbb{R}^n$. Use Proposition 1 to show

$$\langle A\vec{v}, \vec{w} \rangle = \left\langle \vec{v}, A^T \vec{w} \right\rangle .$$

Be sure to justify each step.

We have seen that the dot product allows us to measure *lengths* and *angles* using the formulae

$$\langle \vec{v}, \vec{v} \rangle = \|\vec{v}\|^2 \quad \text{and} \quad \langle \vec{v}, \vec{w} \rangle = \|\vec{v}\| \|\vec{w}\| \cos\theta.$$

Definition 1 *If $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map, then we say that A is an* isometry *if A preserves the dot product. That is, for any $\vec{v}, \vec{u} \in \mathbb{R}^n$, we have*

$$\langle \vec{v}, \vec{w} \rangle = \langle A\vec{v}, A\vec{w} \rangle .$$

It's clear from the definition that an isometry preserves the length of every vector and the angle between any two vectors. A closely related definition is

Definition 2 *An $n \times n$ matrix A is* orthogonal $\iff$ $A^T = A^{-1}$ $\iff$ $A^T A = I$ *or* $AA^T = I$.

2. (10 points) Show that a linear map A is an isometry $\iff$ A is an orthogonal matrix.
Use 11 and 12 and indicate where you use them!

3. (10 points) It's not surprising that if we take an isometry (a map which preserves lengths and angles) and follow it by another isometry (which also preserves lengths and angles), the combination of two maps ... preserves lengths and angles. But there's still a little bit of work to do to verify this in matrix language.

   (1) Suppose that A and B are orthogonal $n \times n$ matrices.
   Prove that AB is an $n \times n$ orthogonal matrix.

   (2) (5 points) Suppose that A is an orthogonal $n \times n$ matrix.
   Prove that A^{-1} is an orthogonal $n \times n$ matrix.