Going backwards - what determines shape?

We have seen that

isometry $\Rightarrow$ same Gauss curvature

same (constant) positive Gauss curvature $\Rightarrow$ isometric

Alas, we will show in homework that

$$x(u,v) = (u\cos v,\ u\sin v,\ v)$$
$$y(u,v) = (u\cos v,\ u\sin v,\ \ln u)$$

have the same Gauss curvature, but different I_p forms. (and hence are not isometric).

Q. What is "enough" information to "recover" the geometry of the surface?

Let's start with the strongest possible interpretation of "recover", and "enough". Recover means "determine up to congruence" and "enough" means isometry + more.

Bonnet Theorem.

If M and M^* are parametrized surfaces with $I_p = I_p^*$ and $II_p = \pm II_p^*$, then M and M^* are <u>congruent</u> (related by a rigid motion of $\mathbb{R}^3$).

Proof. The proof is not very enlightening, but here goes.

Suppose we have curves $\vec{\alpha}, \vec{\alpha}^*$ defined on $[0,b]$ and bases $\vec{v}_i, \vec{v}_i^*$ for $\mathbb{R}^3$ defined on $[0,b]$ so that there is a common set of functions g_{ij}, p_i, and q_{ij} so that

$$\vec{\alpha}'(t) = \sum p_i(t)\vec{v}_i(t), \quad \vec{\alpha}^{*\prime}(t) = \sum p_i(t)\vec{v}_i^*(t)$$

$$\vec{v}_j'(t) = \sum q_{ij}\vec{v}_i(t), \quad \vec{v}_i^{*\prime}(t) = \sum q_{ij}(t)\vec{v}_i^*(t)$$

and $\langle v_i(t), v_j(t)\rangle = g_{ij}(t) = \langle \vec{v}_i^*(t), \vec{v}_j^*(t)\rangle$.

We claim that if $\vec{\alpha}(0)=\vec{\alpha}^*(0)$ and $\vec{v}_i(0)=\vec{v}_i^*(0)$ then $\vec{\alpha}(t)=\vec{\alpha}^*(t)$ and $\vec{v}_i(t)=\vec{v}_i^*(t)$ for all $t\in[0,b]$.

If we let $M(t)=\begin{bmatrix} \uparrow & \uparrow & \uparrow \\ v_1(t) & v_2(t) & v_3(t) \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$ ~~then~~ and $M^*(t)=$ the corresponding matrix of $\vec{v}_i^*$, then

$$M'(t) = \cancel{Q(t)M(t)} = M(t)\,Q(t), \quad M^{*\prime}(t) = \cancel{Q(t)M^*(t)} = M^*(t)\,Q(t).$$

and $M(t)^T M(t) = G(t)$. ~~Let's dif~~ Since the $\vec{v}_i$ form a basis for $\mathbb{R}^3$, $\det M(t)\neq 0$, so this means $\det G(t) = (\det M(t))^2 \neq 0$, and G is invertible. So

$$G(t)\,G^{-1}(t) = I$$

Differentiating,

$$G'(t)\,G^{-1}(t) + G(t)(G^{-1})'(t) = 0.$$

We can solve this for $\cancel{G(t)}\,(G^{-1})'(t)$:

$$G(t)\,(G^{-1})'(t) = -G'(t)\,G^{-1}(t)$$

and

$$(G^{-1})'(t) = -G^{-1}(t)\,G'(t)\,G^{-1}(t).$$

Further, differentiating $G(t) = M(t)^T M(t)$ gives

$$G'(t) = M'(t)^T M(t) + M(t)^T M'(t)$$

$$= (M(t)Q(t))^T M(t) + M^T(t) M(t) Q(t)$$

$$= Q^T(t)\underbrace{M^T(t)M(t)}_{G(t)} + \underbrace{M^T(t)M(t)}_{G(t)}\,Q(t)$$

$$= Q^T(t)\,G(t) + G(t)\,Q(t).$$

Now consider $(M^*G^{-1}M^T)(t)$. We claim it's constant! To see this, we plug and chug

$$(M^*G^{-1}M^T)' = (M^*)'G^{-1}M^T + M^*(G^{-1})'M^T + M^*G^{-1}(M^T)'$$

$$= M^*QG^{-1}M^T - M^*(G^{-1}\underset{\downarrow}{G'}G^{-1})M^T + M^*G^{-1}Q^TM^T$$

$$Q^TG + GQ$$

cancels

$$= M^*QG^{-1}M^T - M^*G^{-1}Q^T\cancel{GG^{-1}}M^T - M^*\cancel{G^{-1}G}QG^{-1}M^T = 0$$

$$+ M^*G^{-1}Q^TM^T$$

cancels

Now at $t=0$, we have $M(0)=M^*(0)$, so

$$(M^* G^{-1} M^T)(0) = (\cancel{M M^{-1}} \cancel{(M^T)^{-1}} M^T)(0)$$
$$= I,$$

and thus

$$M^*(t)\, G^{-1}(t)\, M^T(t) \equiv I$$

Since as before, $G^{-1}(t) = M^{-1}(t)(M^{-1})^T(t)$, we have

$$M^*(t)\, M^{-1}(t) \cancel{(M^T)^{-1}(t)\, M^T(t)} \equiv I$$

and

$$M^*(t)\, M^{-1}(t) \equiv I.$$

Thus $M^*(t) \equiv M(t)$. Since this means $\vec{v}_i(t) = \vec{v}_i^{\,*}(t)$, it means $\vec{\alpha}'(t) = \vec{\alpha}^{*\prime}(t)$ as well, so since $\vec{\alpha}(0) = \vec{\alpha}^*(0)$, we have $\vec{\alpha}(t) = \vec{\alpha}'(t)$.

Now we just observe that for any points p, p^*, we have that

$$\begin{bmatrix} x_u & x_v & n \end{bmatrix} = Q \begin{bmatrix} x_u^* & x_v^* & \vec{n}^* \end{bmatrix}$$

~~are related by a~~ for Q in $SO(3)$.

Miniproof. Since both matrices are invertible (call them M, M^*), some such Q exists:

$$Q = M(M^*)^{-1}$$

Now

$$\begin{aligned} QQ^T &= M(M^*)^{-1}{(M^*)^{-1}}^T M^T \\ &= M(M^{*T}M^*)^{-1}M^T \\ &= M\left(\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1}\right)M^T \end{aligned}$$

But

$$M^TM = \begin{bmatrix} E & F \\ F & G \end{bmatrix}, \text{ so } M^T = \begin{bmatrix} E & F \\ F & G \end{bmatrix} M^{-1} \text{ and}$$

$$= M \cancel{\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1}} \cancel{\begin{bmatrix} E & F \\ F & G \end{bmatrix}} M^{-1} = MM^{-1} = I.$$

so by a rigid motion we can arrange that $(\vec{x}_u, \vec{x}_v, \vec{n}) = (\vec{x}^*_u, \vec{x}^*_v, \vec{n}^*)$.

But now the lemma applies: ~~along~~ if we pick a path in coordinate space (the u-v plane) from u_0 with $x(u_0) = p$ to any u_1, then along the path,

$$\left.\begin{array}{l} x_u' = \\ x_v' = \\ \vec{n}' = \end{array}\right\} \begin{array}{l} \text{some combination of } x_{uu} \text{ and } x_{uv} \\ \text{and } n_u, n_v \text{, hence written in} \\ x_u, x_v, n \text{ basis in terms of} \\ \Gamma^k_{ij}, l, m, n \text{ and so determined by} \\ I_p \text{ and } II_p \end{array}$$

Further $\alpha' =$ a linear combination of x_u and x_v, so our ~~theorem~~ claim works.

Thus $\alpha(b) = p = \alpha^*(b) = p^*$ and the surfaces are identical. $\square$

Now that's Kind of expected, but something more surprising is a theorem of ~~Chern~~. Minkowski (eventually most powerfully proved by Chern)

Theorem [Chern, 1957] If S and S^* are ~~of~~ convex surfaces of class C^2 and the ~~a~~ map ~~from~~ $f: S \to S^*$ so that $\vec{n}(p) = \vec{n}^*(f(p))$ has $K(p) = K^*(f(p)) > 0$, then S and S^* are congruent.

This is similar in spirit to Cohn-Vossen's rigidity theorem:

Theorem. [CV-1927] If M and M^* are closed convex surfaces in $\mathbb{R}^3$ and M, M^* are isometric, then M, M^* are congruent.

So

corresponding nonconstant positive curvature $\Rightarrow$ congruent.

And I really like

Theorem [Lawson-Tribuzy, 1981]

If M and M^* are (topological) spheres with nonconstant mean curvature so that M, M^* are isometric and have the same mean curvature, then M, M^* are congruent.

If M is a compact surface with nonconstant mean curvature, it has at most two isometric embeddings into $\mathbb{R}^3$.

For constant mean curvature surfaces, such as minimal surfaces, everything falls apart and there many be entire families of isometric embeddings. (!)