The Second Fundamental Form II

Now that we have that the shape operator is symmetric, we can use it to define a new inner product (quadratic form)

$$\mathrm{II}_p(\vec{u},\vec{v}) = \langle \vec{u},\vec{v} \rangle_{\mathrm{II}_p}$$
$$= \langle \vec{u}, S_p(\vec{v}) \rangle_{\mathbb{R}^3}$$

We can write out II_p in the x_u, x_v basis as a matrix. Let

$$\vec{u} = a\vec{x}_u + b\vec{x}_v$$
$$\vec{v} = c\vec{x}_u + d\vec{x}_v$$

Then we have

$$\mathrm{II}_p(\vec{u},\vec{v}) = \left\langle \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} \ell & m \\ m & n \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \right\rangle_{\mathbb{R}^2}$$

$$= \ell ac + m(bc+ad) + nbd$$

and we see by plugging in 1's and 0's for a, b, c, d that

$$\ell = \mathrm{II}_p(\vec{X}_u,\vec{X}_u) = -\langle D_{X_u} n, X_u\rangle = \langle \vec{n}, \vec{X}_{uu}\rangle$$

$$m = \mathrm{II}_p(\vec{X}_u,\vec{X}_v) = -\langle D_{X_u} n, X_v\rangle = \langle \vec{n}, \vec{X}_{uv}\rangle$$

$$n = \mathrm{II}_p(\vec{X}_v,\vec{X}_v) = -\langle D_{X_v} n, X_v\rangle = \langle \vec{n}, \vec{X}_{vv}\rangle$$

We note that this is <u>not</u> the matrix for $S_p(-)$ as a linear operator in the X_u, X_v basis.

Because

$$\mathrm{II}_p(\vec{u},\vec{v}) = \langle \vec{u}, S_p(\vec{v}) \rangle_{\mathbb{R}^3}$$

$$= \left\langle \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} S_p \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \right\rangle_{\mathbb{R}^2}$$

we see that

$$\begin{bmatrix} l & m \\ m & n \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} S_p \end{bmatrix}$$

and

$$\begin{bmatrix} S_p \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} l & m \\ m & n \end{bmatrix}.$$

Since

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} = \frac{1}{EG-F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix}$$

we can see $[S_p]$ is a symmetric matrix. $\Leftrightarrow \begin{bmatrix} l & m \\ m & n \end{bmatrix} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} = \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} l & m \\ m & n \end{bmatrix}$

which is not always true!

Definition. Since S_p is a 2×2 symmetric matrix it has two real eigenvalues $K_1(p)$ and $K_2(p)$.

1) The eigenvalues are called principal curvatures.

2) The corresponding eigenvectors are called principal directions.

3) A curve α in M is called a line of curvature if α' always points in a principal direction.

Recall that if $K_1 \neq K_2$ the principal directions are orthogonal

$$\Big(\langle v_1, S_p(v_2) \rangle = \langle v_1, \lambda_2 v_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle$$

$$\langle S_p(v_1), v_2 \rangle = \langle \lambda_1 v_1, v_2 \rangle = \lambda_1 \langle v_1, v_2 \rangle$$

so $\lambda_1 = \lambda_2$ or $\langle v_1, v_2 \rangle = 0$.)

~~so~~ and if $\lambda_1 = \lambda_2$ then all vectors are eigenvectors and all directions are principal.

Thus we can always pick an ~~the~~ orthonormal basis of principal directions v_1, v_2. With respect to such a basis,

Proposition. If $\vec{v} = \cos\theta\,\vec{v}_1 + \sin\theta\,\vec{v}_2$ then the slice curvature $\mathrm{II}_p(\vec{v},\vec{v}) = K_1\cos^2\theta + K_2\sin^2\theta$.

Proof.

$$\mathrm{II}_p(\vec{v},\vec{v}) = \langle \vec{v}, S_p(\vec{v}) \rangle$$

$$= \langle \cos\theta\,\vec{v}_1 + \sin\theta\,\vec{v}_2,\ \lambda_1\cos\theta\, v_1 + \lambda_2\cos\theta\, v_2 \rangle$$

$$= \lambda_1\cos^2\theta + \lambda_2\sin^2\theta.$$

$$= K_1\cos^2\theta + K_2\sin^2\theta. \quad \square$$

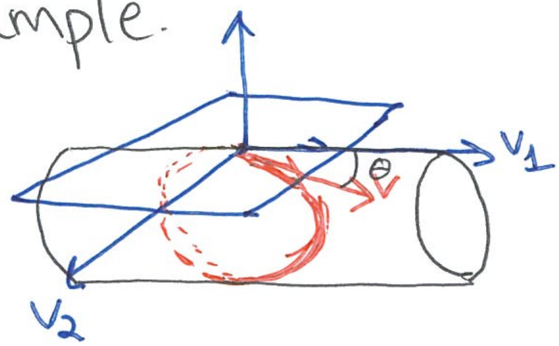
This lets us draw a nice conclusion that's often helpful in computations.

The principal directions are the directions with largest and smallest slice curvatures, (values of $\mathrm{II}_p(\vec{v},\vec{v})$).

Proof. Differentiate $K_1\cos^2\theta + K_2\sin^2\theta$ w.r.t. θ and find the critical points are at multiples of $\pi/2$ if $K_1 \neq K_2$ (and everywhere if $K_1 = K_2$). $\square$

We can use this to find the principal directions.

Example.



Cutting a cylinder at an angle θ yields an ellipse. The curvature at the top is given by

$$K(p) = \mathrm{II}_p(\vec{v}, \vec{v})$$

$$= K_1 \cos^2\theta + K_2 \sin^2\theta$$

$$= 0 \cos^2\theta + K_2 \sin^2\theta$$

curvature of line ∥ to axis

curvature of circular cross-section

Definition. If the slice curvature $\mathrm{II}_p(\vec{v},\vec{v}) = 0$, we say $\vec{v}$ is an <u>asymptotic direction</u>.

A curve α is an asymptotic curve $\Leftrightarrow$ α' is always in an asymptotic direction.

Proposition. There is an asymptotic direction $\Leftrightarrow$ $K_1 K_2 < 0$.

Proof. The slice curvatures $K_1 \cos^2\theta + K_2 \sin^2\theta$ interpolate between K_1 and K_2. $\square$

We now define

Definition. The determinant $\det S_p = k_1 k_2 = K$ is called the <u>Gauss</u> <u>curvature</u> of M at p. The ~~trace~~ number $\frac{1}{2}(k_1 + k_2) = \frac{1}{2} \operatorname{trace} S_p = H$ is called the <u>mean</u> <u>curvature</u>.

We say

$K = 0 \iff M$ is flat

$H = 0 \iff M$ is a minimal surface.