The Second Fundamental Form II

Now that we have that the shape operator is symmetric, we can use it to define a new inner product (quadratic form)

\[ \hat{\Pi}_p(\hat{u}, \hat{v}) = \langle \hat{u}, \hat{v} \rangle_{\hat{\Pi}_p} = \langle \hat{u}, S_p(\hat{v}) \rangle_{\mathbb{R}^3} \]

We can write out \( \hat{\Pi}_p \) in the \( x_u, x_v \) basis as a matrix. Let

\[ \hat{u} = a \hat{x}_u + b \hat{x}_v \]
\[ \hat{v} = c \hat{x}_u + d \hat{x}_v \]
Then we have

\[ \Pi_p(\hat{\mathbf{u}}, \hat{\mathbf{v}}) = \left\langle \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} l & m \\ m & n \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \right\rangle_{\mathbb{R}^2} \]

\[ = lac + m(bc+ad) + nbd \]

and we see by plugging in 1's and 0's for \(a, b, c, d\) that

\[ l = \Pi_p(\hat{x}_u, \hat{x}_u) = -\left\langle D_{x_u} \hat{n}, x_u \right\rangle = \left\langle \hat{n}, \hat{x}_{uu} \right\rangle \]

\[ m = \Pi_p(\hat{x}_u, \hat{x}_v) = -\left\langle D_{x_u} \hat{n}, x_v \right\rangle = \left\langle \hat{n}, \hat{x}_{uv} \right\rangle \]

\[ n = \Pi_p(\hat{x}_v, \hat{x}_v) = -\left\langle D_{x_v} \hat{n}, x_v \right\rangle = \left\langle \hat{n}, \hat{x}_{vv} \right\rangle \]

We note that this is not the matrix for \(S_p(-)\) as a linear operator in the \(x_u, x_v\) basis.
Because

$$\Pi_p(\vec{u}, \vec{v}) = \left\langle \vec{u}, S_p(\vec{v}) \right\rangle \in \mathbb{R}^3$$

$$= \left\langle \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} S_p \\ \vdots \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \right\rangle \in \mathbb{R}^2$$

we see that

$$\begin{bmatrix} l & m \\ m & n \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} S_p \end{bmatrix}$$

and

$$\begin{bmatrix} S_p \end{bmatrix} = \begin{bmatrix} E & F \end{bmatrix}^{-1} \begin{bmatrix} l & m \\ m & n \end{bmatrix}.$$

Since

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} = \frac{1}{EG-F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix}$$

we can see $$\begin{bmatrix} S_p \end{bmatrix}$$ is a symmetric matrix. $$\iff \begin{bmatrix} l & m \\ m & n \end{bmatrix} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} = \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} l & m \\ m & n \end{bmatrix}$$

which is not always true.
Definition. Since $S_p$ is a $2 \times 2$ symmetric matrix it has two real eigenvalues $k_1(p)$ and $k_2(p)$.

1) The eigenvalues are called principal curvatures.

2) The corresponding eigenvectors are called principal directions.

3) A curve $\alpha$ in $M$ is called a line of curvature if $\alpha'$ always points in a principal direction.

Recall that if $k_1 \neq k_2$ the principal directions are orthogonal

\[
\begin{align*}
\langle v_1, S_p(v_2) \rangle &= \lambda_1 \langle v_1, v_2 \rangle = \lambda_1 \langle v_1, v_2 \rangle \\
\langle S_p(v_1), v_2 \rangle &= \langle \lambda_1 v_1, v_2 \rangle = \lambda_1 \langle v_1, v_2 \rangle
\end{align*}
\]
so \( \lambda_1 = \lambda_2 \) or \( \langle v_1, v_2 \rangle = 0 \).

And if \( \lambda_1 = \lambda_2 \) then all vectors are eigenvectors and all directions are principal.

Thus we can always pick an orthonormal basis of principal directions \( v_1, v_2 \). With respect to such a basis, Proposition. If \( \hat{V} = \cos \Theta \hat{v}_1 + \sin \Theta \hat{v}_2 \) then the slice curvature \( \Pi_p (\hat{V}, \hat{V}) = k_1 \cos^2 \Theta + k_2 \sin^2 \Theta \).

Proof.

\[
\Pi_p (\hat{V}, \hat{V}) = \langle \hat{V}, S_p (\hat{V}) \rangle \\
= \langle \cos \Theta \hat{v}_1 + \sin \Theta \hat{v}_2, \lambda_1 \cos \Theta \hat{v}_1 + \lambda_2 \cos \Theta \hat{v}_2 \rangle \\
= \lambda_1 \cos^2 \Theta + \lambda_2 \sin^2 \Theta. \quad \Box \\
= k_1 \cos^2 \Theta + k_2 \sin^2 \Theta, \quad \Box
\]
This lets us draw a nice conclusion that's often helpful in computations.

The principal directions are the directions with largest and smallest slice curvatures \( \Pi_p(v,v) \) (values of \( \Pi_p(v,v) \)).

Proof. Differentiate \( k_1 \cos^2 \theta + k_2 \sin^2 \theta \) w.r.t. \( \theta \) and find the critical points are at multiples of \( \pi/2 \) if \( k_1 \neq k_2 \) (and everywhere if \( k_1 = k_2 \)). \( \square \)

We can use this to find the principal directions.
Example.

Cutting a cylinder at an angle $\theta$ yields an ellipse. The curvature at the top is given by

$$K(p) = II_p(\vec{v}, \vec{v})$$

$$= \frac{k_1}{a^2} \cos^2 \theta + \frac{k_2}{b^2} \sin^2 \theta$$

$$= 0 \cos^2 \theta + k_2 \sin^2 \theta$$

Curvature of line $\Pi$ to axis

Curvature of circular cross-section
Definition. If the slice curvature $\Pi_p(\hat{\nu},\hat{\nu}) = O$, we say $\hat{\nu}$ is an asymptotic direction.

A curve $\alpha$ is an asymptotic curve $\iff \alpha'$ is always in an asymptotic direction.

Proposition. There is an asymptotic direction $\iff k_1 k_2 < 0$.

Proof. The slice curvatures $k_1 \cos^2 \Theta + k_2 \sin^2 \Theta$ interpolate between $k_1$ and $k_2$. $\Box$
We now define

Definition. The determinant \( \det S_p = k_1 k_2 = k \) is called the Gauss curvature of \( M \) at \( p \). The trace number \( \frac{1}{2} (k_1 + k_2) = \frac{1}{2} \text{trace} S_p \) is called the mean curvature. \( \nabla H \)

We say

\[ K = 0 \iff M \text{ is flat} \]
\[ H = 0 \iff M \text{ is a minimal surface} \]