Alternatively, since \( \tan(\theta/2) = e^t \), we have
\[
\sin \theta = 2 \sin(\theta/2) \cos(\theta/2) = \frac{2e^t}{1 + e^{2t}} = \frac{2}{e^t + e^{-t}} = \text{sech} \ t
\]
\[
\cos \theta = \cos^2(\theta/2) - \sin^2(\theta/2) = \frac{1 - e^{2t}}{1 + e^{2t}} = \frac{e^{-t} - e^t}{e^t + e^{-t}} = -\tanh t,
\]
and so we can parametrize the tractrix instead by
\[
\beta(t) = (t - \tanh t, \tanh t), \quad t \geq 0. \quad \nabla
\]

The fundamental concept underlying the geometry of curves is the arclength of a parametrized curve.

**Definition.** If \( \alpha: [a, b] \rightarrow \mathbb{R}^3 \) is a parametrized curve, then for any \( a \leq t \leq b \), we define its arclength from \( a \) to \( t \) to be \( s(t) = \int_a^t \|\alpha'(u)\| \, du \). That is, the distance a particle travels—the arclength of its trajectory—is the integral of its speed.

An alternative approach is to start with the following

**Definition.** Let \( \alpha: [a, b] \rightarrow \mathbb{R}^3 \) be a (continuous) parametrized curve. Given a partition \( \mathcal{P} = \{a = t_0 < t_1 < \cdots < t_k = b\} \) of the interval \([a, b]\), let
\[
\ell(\alpha, \mathcal{P}) = \sum_{i=1}^k \|\alpha(t_i) - \alpha(t_{i-1})\|.
\]
That is, \( \ell(\alpha, \mathcal{P}) \) is the length of the inscribed polygon with vertices at \( \alpha(t_i), i = 0, \ldots, k \), as indicated in

Given this partition, \( \mathcal{P} \), of \([a, b]\),

![Diagram](https://example.com/diagram.png)

**Figure 1.10**

Figure 1.10. We define the arclength of \( \alpha \) to be
\[
\text{length}(\alpha) = \sup\{\ell(\alpha, \mathcal{P}) : \mathcal{P} \text{ a partition of } [a, b]\},
\]
provided the set of polygonal lengths is bounded above.

Now, using this definition, we can prove that the distance a particle travels is the integral of its speed. We will need to use the result of Exercise A.2.4.
Proposition 1.1. Let \( \alpha : [a, b] \rightarrow \mathbb{R}^3 \) be a piecewise-\( C^1 \) parametrized curve. Then

\[
\text{length}(\alpha) = \int_a^b \|\alpha'(t)\| \, dt.
\]

Proof. For any partition \( P \) of \([a, b]\), we have

\[
\ell(\alpha, P) = \sum_{i=1}^{k} \|\alpha(t_i) - \alpha(t_{i-1})\| = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \|\alpha'(t)\| \, dt \leq \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \|\alpha'(t)\| \, dt = \int_a^b \|\alpha'(t)\| \, dt,
\]

so \( \text{length}(\alpha) \leq \int_a^b \|\alpha'(t)\| \, dt \). The corresponding inequality holds on any interval.

Now, for \( a \leq t \leq b \), define \( s(t) \) to be the arclength of the curve \( \alpha \) on the interval \([a, t]\). Then for \( h > 0 \) we have

\[
\frac{\|\alpha(t+h) - \alpha(t)\|}{h} \leq \frac{s(t+h) - s(t)}{h} \leq \frac{1}{h} \int_t^{t+h} \|\alpha'(u)\| \, du,
\]

since \( s(t+h) - s(t) \) is the arclength of the curve \( \alpha \) on the interval \([t, t+h]\). (See Exercise 8 for the first inequality and the first paragraph for the second.) Now

\[
\lim_{h \to 0^+} \frac{\|\alpha(t+h) - \alpha(t)\|}{h} = \|\alpha'(t)\| = \lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \|\alpha'(u)\| \, du.
\]

Therefore, by the squeeze principle,

\[
\lim_{h \to 0^+} \frac{s(t+h) - s(t)}{h} = \|\alpha'(t)\|.
\]

A similar argument works for \( h < 0 \), and we conclude that \( s'(t) = \|\alpha'(t)\| \). Therefore,

\[
s(t) = \int_a^t \|\alpha'(u)\| \, du, \quad a \leq t \leq b,
\]

and, in particular, \( s(b) = \text{length}(\alpha) = \int_a^b \|\alpha'(t)\| \, dt \), as desired. \( \square \)

If \( \|\alpha'(t)\| = 1 \) for all \( t \in [a, b] \), i.e., \( \alpha \) always has speed 1, then \( s(t) = t - a \). We say the curve \( \alpha \) is parametrized by arclength if \( s(t) = t \) for all \( t \). In this event, we usually use the parameter \( s \in [0, L] \) and write \( \alpha(s) \).

Example 3. (a) Let \( \alpha(t) = (\frac{1}{2}(1 + t)^{3/2}, \frac{1}{3}(1 - t)^{3/2}, \frac{1}{\sqrt{2}}t), t \in (-1, 1) \). Then we have \( \alpha'(t) = (\frac{1}{2}(1 + t)^{1/2}, -\frac{1}{2}(1 - t)^{1/2}, \frac{1}{\sqrt{2}}) \), and \( \|\alpha'(t)\| = 1 \) for all \( t \). Thus, \( \alpha \) always has speed 1.

(b) The standard parametrization of the circle of radius \( a \) is \( \alpha(t) = (a \cos t, a \sin t), t \in [0, 2\pi] \), so \( \alpha'(t) = (-a \sin t, a \cos t) \) and \( \|\alpha'(t)\| = a \). It is easy to see from the chain rule that if we reparametrize the curve by \( \beta(s) = (a \cos(s/a), a \sin(s/a)), s \in [0, 2\pi a], \) then \( \beta'(s) = (-\sin(s/a), a \cos(s/a)) \) and \( \|\beta'(s)\| = 1 \) for all \( s \). Thus, the curve \( \beta \) is parametrized by arclength. \( \triangledown \)
An important observation from a theoretical standpoint is that any regular parametrized curve can be reparametrized by arclength. For if \( \alpha \) is regular, the arclength function \( s(t) = \int_a^t \| \alpha'(u) \| du \) is an increasing differentiable function (since \( s'(t) = \| \alpha'(t) \| > 0 \) for all \( t \)), and therefore has a differentiable inverse function \( t = t(s) \). Then we can consider the parametrization

\[
\beta(s) = \alpha(t(s)).
\]

Note that the chain rule tells us that

\[
\beta'(s) = \alpha'(t(s))t'(s) = \alpha'(t(s))/s'(t(s)) = \alpha'(t(s))/\|\alpha'(t(s))\|
\]
is everywhere a unit vector; in other words, \( \beta \) moves with speed 1.

**EXERCISES 1.1**

1. Parametrize the unit circle (less the point \((-1,0)\)) by the length \( t \) indicated in Figure 1.11.

![Figure 1.11](image)

2. Consider the helix \( \alpha(t) = (a \cos t, a \sin t, bt) \). Calculate \( \alpha'(t), \|\alpha'(t)\| \), and reparametrize \( \alpha \) by arclength.

3. Let \( \alpha(t) = \left( \frac{1}{\sqrt{3}} \cos t + \frac{1}{\sqrt{2}} \sin t, \frac{1}{\sqrt{3}} \cos t - \frac{1}{\sqrt{2}} \sin t \right) \). Calculate \( \alpha'(t), \|\alpha'(t)\| \), and reparametrize \( \alpha \) by arclength.

4. Parametrize the graph \( y = f(x), a \leq x \leq b \), and show that its arclength is given by the traditional formula

\[
\text{length} = \int_a^b \sqrt{1 + (f'(x))^2} \, dx.
\]

5. a. Show that the arclength of the catenary \( \alpha(t) = (t, \cosh t) \) for \( 0 \leq t \leq b \) is \( \sinh b \).

   b. Reparametrize the catenary by arclength. (Hint: Find the inverse of \( \sinh \) by using the quadratic formula.)

6. Consider the curve \( \alpha(t) = (e^t, e^{-t}, \sqrt{2}t) \). Calculate \( \alpha'(t), \|\alpha'(t)\| \), and reparametrize \( \alpha \) by arclength, starting at \( t = 0 \).