Remark. H. Hopf proved a stronger result, which requires techniques from complex analysis: If M is a compact surface topologically equivalent to a sphere and having constant *mean* curvature, then M must be a sphere.

We conclude this section with the analogue of Theorem 3.1 of Chapter 1.

Theorem 3.8 (Fundamental Theorem of Surface Theory)**.** Uniqueness: *Two parametrized surfaces* $\mathbf{x}, \mathbf{x}^*: U \to \mathbb{R}^3$ *are congruent (i.e., differ by a rigid motion) if and only if* $\mathrm{I} = \mathrm{I}^*$ *and* $\mathrm{II} = \pm\mathrm{II}^*$. Existence: *Moreover, given differentiable functions* E, F, G, ℓ, m, *and* n *with* $E > 0$ *and* $EG - F^2 > 0$ *and satisfying the Codazzi and Gauss equations, there exists (locally) a parametrized surface* $\mathbf{x}(u, v)$ *with the respective* I *and* II.

Proof. The existence statement requires some theorems from partial differential equations beyond our reach at this stage. The uniqueness statement, however, is much like the proof of Theorem 3.1 of Chapter 1. (The main technical difference is that we no longer are lucky enough to be working with an *orthonormal* basis at each point, as we were with the Frenet frame.)

First, suppose $\mathbf{x}^* = \Psi\circ\mathbf{x}$ for some rigid motion $\Psi: \mathbb{R}^3 \to \mathbb{R}^3$ (i.e., $\Psi(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for some $\mathbf{b} \in \mathbb{R}^3$ and some 3×3 orthogonal matrix A). Since a translation doesn't change partial derivatives, we may assume that $\mathbf{b} = \mathbf{0}$. Now, since orthogonal matrices preserve length and dot product, we have $E^* = \|\mathbf{x}_u^*\|^2 = \|A\mathbf{x}_u\|^2 = \|\mathbf{x}_u\|^2 = E$, etc., so $\mathrm{I} = \mathrm{I}^*$. If $\det A > 0$, then $\mathbf{n}^* = A\mathbf{n}$, whereas if $\det A < 0$, then $\mathbf{n}^* = -A\mathbf{n}$. Thus, $\ell^* = \mathbf{x}_{uu}^* \cdot \mathbf{n}^* = A\mathbf{x}_{uu} \cdot (\pm A\mathbf{n}) = \pm\ell$, the positive sign holding when $\det A > 0$ and the negative when $\det A < 0$. Thus, $\mathrm{II}^* = \mathrm{II}$ if $\det A > 0$ and $\mathrm{II}^* = -\mathrm{II}$ if $\det A < 0$.

Conversely, suppose $\mathrm{I} = \mathrm{I}^*$ and $\mathrm{II} = \pm\mathrm{II}^*$. By composing $\mathbf{x}^*$ with a reflection, if necessary, we may assume that $\mathrm{II} = \mathrm{II}^*$. Now we need the following

Lemma 3.9. *Suppose* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\alpha}^*$ *are smooth functions on* $[0, b]$, $\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3$ *and* $\mathbf{v}_1^*\mathbf{v}_2^*\mathbf{v}_3^*$ *are smoothly varying bases for* $\mathbb{R}^3$, *also defined on* $[0, b]$, *so that*

$$\mathbf{v}_i(t)\cdot\mathbf{v}_j(t) = \mathbf{v}_i^*(t)\cdot\mathbf{v}_j^*(t) = g_{ij}(t), \qquad i, j = 1, 2, 3,$$

$$\boldsymbol{\alpha}'(t) = \sum_{i=1}^{3} p_i(t)\mathbf{v}_i(t) \qquad \text{and} \qquad \boldsymbol{\alpha}^{*\prime}(t) = \sum_{i=1}^{3} p_i(t)\mathbf{v}_i^*(t),$$

$$\mathbf{v}_j'(t) = \sum_{i=1}^{3} q_{ij}\mathbf{v}_i(t) \qquad \text{and} \qquad \mathbf{v}_j^{*\prime}(t) = \sum_{i=1}^{3} q_{ij}\mathbf{v}_i^*(t), \quad j = 1, 2, 3.$$

(Note that the coefficient functions p_i *and* q_{ij} *are the same for both the starred and unstarred equations.) If* $\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}^*(0)$ *and* $\mathbf{v}_i(0) = \mathbf{v}_i^*(0)$, $i = 1, 2, 3$, *then* $\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}^*(t)$ *and* $\mathbf{v}_i(t) = \mathbf{v}_i^*(t)$ *for all* $t \in [0, b]$, $i = 1, 2, 3$.

Fix a point $\mathbf{u}_0 \in U$. By composing $\mathbf{x}^*$ with a rigid motion, we may assume that *at* $\mathbf{u}_0$ we have $\mathbf{x} = \mathbf{x}^*$, $\mathbf{x}_u = \mathbf{x}_u^*$, $\mathbf{x}_v = \mathbf{x}_v^*$, and $\mathbf{n} = \mathbf{n}^*$ (why?). Choose an arbitrary $\mathbf{u}_1 \in U$, and join $\mathbf{u}_0$ to $\mathbf{u}_1$ by a path $\mathbf{u}(t)$, $t \in [0, b]$, and apply the lemma with $\boldsymbol{\alpha} = \mathbf{x}\circ\mathbf{u}$, $\mathbf{v}_1 = \mathbf{x}_u\circ\mathbf{u}$, $\mathbf{v}_2 = \mathbf{x}_v\circ\mathbf{u}$, $\mathbf{v}_3 = \mathbf{n}\circ\mathbf{u}$, $p_i = u_i'$, and the q_{ij} prescribed by the equations (†) and (††). Since $\mathrm{I} = \mathrm{I}^*$ and $\mathrm{II} = \mathrm{II}^*$, the same equations hold for $\boldsymbol{\alpha}^* = \mathbf{x}^*\circ\mathbf{u}$, and so $\mathbf{x}(\mathbf{u}_1) = \mathbf{x}^*(\mathbf{u}_1)$ as desired. That is, the two parametrized surfaces are identical. $\square$

Proof of Lemma 3.9. Introduce the matrix function of t

$$M(t) = \begin{bmatrix} | & | & | \\ \mathbf{v}_1(t) & \mathbf{v}_2(t) & \mathbf{v}_3(t) \\ | & | & | \end{bmatrix},$$

and analogously for $M^*(t)$. Then the displayed equations in the statement of the Lemma can be written as

$$M'(t) = M(t)Q(t) \qquad \text{and} \qquad M^{*\prime}(t) = M^*(t)Q(t).$$

On the other hand, we have $M(t)^{\mathsf{T}}M(t) = G(t)$. Since the $\mathbf{v}_i(t)$ form a basis for $\mathbb{R}^3$ for each t, we know the matrix G is invertible. Now, differentiating the equation $G(t)G^{-1}(t) = I$ yields $(G^{-1})'(t) = -G^{-1}(t)G'(t)G^{-1}(t)$, and differentiating the equation $G(t) = M(t)^{\mathsf{T}}M(t)$ yields $G'(t) = M'(t)^{\mathsf{T}}M(t) + M(t)^{\mathsf{T}}M'(t) = Q(t)^{\mathsf{T}}G(t) + G(t)Q(t)$. Now consider

$$\begin{aligned}
(M^*G^{-1}M^{\mathsf{T}})'(t) &= M^{*\prime}(t)G(t)^{-1}M(t)^{\mathsf{T}} + M^*(t)(G^{-1})'(t)M(t)^{\mathsf{T}} + M^*(t)G(t)^{-1}M'(t)^{\mathsf{T}} \\
&= M^*(t)Q(t)G(t)^{-1}M(t)^{\mathsf{T}} + M^*(t)\big(-G(t)^{-1}G'(t)G(t)^{-1}\big)M(t)^{\mathsf{T}} \\
&\qquad + M^*(t)G(t)^{-1}Q(t)^{\mathsf{T}}M(t)^{\mathsf{T}} \\
&= M^*(t)Q(t)G(t)^{-1}M(t)^{\mathsf{T}} - M^*(t)G(t)^{-1}Q(t)^{\mathsf{T}}M(t)^{\mathsf{T}} - M^*(t)Q(t)G(t)^{-1}M(t)^{\mathsf{T}} \\
&\qquad + M^*(t)G(t)^{-1}Q(t)^{\mathsf{T}}M(t)^{\mathsf{T}} = \mathrm{O}.
\end{aligned}$$

Since $M(0) = M^*(0)$, we have $M^*(0)G(0)^{-1}M(0)^{\mathsf{T}} = M(0)M(0)^{-1}M(0)^{\mathsf{T}-1}M(0)^{\mathsf{T}} = I$, and so $M^*(t)G(t)^{-1}M(t)^{\mathsf{T}} = I$ for all $t \in [0,b]$. It follows that $M^*(t) = M(t)$ for all $t \in [0,b]$, and so $\boldsymbol{\alpha}^{*\prime}(t) - \boldsymbol{\alpha}'(t) = \mathbf{0}$ for all t as well. Since $\boldsymbol{\alpha}^*(0) = \boldsymbol{\alpha}(0)$, it follows that $\boldsymbol{\alpha}^*(t) = \boldsymbol{\alpha}(t)$ for all $t \in [0,b]$, as we wished to establish. $\square$

EXERCISES 2.3

1. Calculate the Christoffel symbols for a cone, $\mathbf{x}(u,v) = (u\cos v, u\sin v, u)$, both directly (as in Example 1) and by using the formulas (‡).
2. Calculate the Christoffel symbols for the following parametrized surfaces. Then check in each case that the Codazzi equations and the first Gauss equation hold.
   a. the plane, parametrized by polar coordinates: $\mathbf{x}(u,v) = (u\cos v, u\sin v, 0)$
   b. a helicoid: $\mathbf{x}(u,v) = (u\cos v, u\sin v, v)$
   ♯c. a cone: $\mathbf{x}(u,v) = (u\cos v, u\sin v, cu)$, $c \neq 0$
   ♯*d. a surface of revolution: $\mathbf{x}(u,v) = \big(f(u)\cos v, f(u)\sin v, g(u)\big)$, with $f'(u)^2 + g'(u)^2 = 1$
3. Use the first Gauss equation to derive the formula $(*)$ given on p. 60 for Gaussian curvature.
4. Check the Gaussian curvature of the sphere using the formula $(*)$ on p. 60.
5. Check that for a parametrized surface with $E = G = \lambda(u,v)$ and $F = 0$, the Gaussian curvature is given by $K = -\dfrac{1}{2\lambda}\nabla^2(\ln\lambda)$. (Here $\nabla^2 f = \dfrac{\partial^2 f}{\partial u^2} + \dfrac{\partial^2 f}{\partial v^2}$ is the Laplacian of f.)
6. Prove there is no *compact* minimal surface $M \subset \mathbb{R}^3$.