Indicatrices, and some Comparisons.

We now know that $k_1, k_2$ or $k_1', k_2'$ completely determine the geometry of a space curve. This means that we ought to want to decode these functions to understand curve geometry.

Here's a helpful construction:

Definition. The tangent indicatrix of $y(s)$ is the spherical curve $T(s)$.
Proposition. $T(s)$ is not arclength-parametrized (unless $y$ has constant curvature); the speed of $T(s)$ is equal to $X(s)$, the length of $T(s)$ is the total curvature $\int x(s) ds$.

Proof. $T'(s) = X(s) N(s)$, so $|T'(s)| = x(s)$, recalling that $x(s) \geq 0$.

If we think of $T(s)$ as a new space curve, it's helpful to frame it by the normal to the sphere.

This is going to lead us into a notational morass, so let's establish conventions now.
Original Curve
\[ y(s) \quad x(s) \]
\[ T(s) \quad \hat{n}(s) \]
\[ N(s) \quad k_\perp(s) \]
\[ B(s) \quad k_\parallel(s) \]
\[ s = \text{arc length parameter} \]

Tangent indicatrix
\[ \hat{y}(s) \quad \hat{x}(s) \]
\[ \hat{T}(s) \quad \hat{n}(s) \]
\[ \hat{F}_\perp(s) \quad \hat{k}_\perp(s) \]
\[ \hat{F}_\parallel(s) \quad \hat{k}_\parallel(s) \]
\[ \hat{s} = \text{arc length parameter} \]

We have so far
\[ \hat{y}(s) = T(s) \]
\[ \hat{T}(s) = N(s) \]
\[ |\hat{y}'(s)| = x(s) \]

Proposition. \( \hat{F}_\perp \) is a Bishop frame, with \( \hat{k}_\perp = 1 \).

Proof. We need only check that
\( \hat{F}_\perp \) is parallel to \( \hat{T} \). But
\[
\frac{d}{ds} F_\perp(s(s)) = \frac{d}{ds} F_\perp(s(s)) \cdot \frac{d}{ds}s
\]
\[
= \frac{d}{ds} T(s) \cdot \frac{d}{ds}s = -x(s)N(s) \cdot \frac{d}{ds}s
\]
\[
\begin{aligned}
&= \left( x(s) \frac{d}{d\tilde{s}} s \right) \tilde{T}(s) \\
& \quad \text{ (scalar)}
\end{aligned}
\]

In fact, we can compute \( \frac{d}{d\tilde{s}} s \) using the fact that \( \frac{d}{d\tilde{s}} \tilde{s} = 1 \). Since \( s(\tilde{s}) \) and \( \tilde{s}(s) \) are inverse functions, this means that \( \frac{d}{d\tilde{s}} s = \frac{1}{k} \) and

\[
\frac{d}{d\tilde{s}} F_1 = \tilde{T}(\tilde{s})
\]

and so \( k_1 = 1 \). \( \Box \)

We now compute \( \tilde{k}_2 \). This is easy if we just realize it's

\[
\left< \tilde{T}, \frac{d}{d\tilde{s}} \tilde{T} \times \tilde{F}_1 \right> = \left< \tilde{T}, \left( \frac{d}{d\tilde{s}} \tilde{T} \right) \times \tilde{F}_1 \right> + \left< \tilde{T}, \tilde{T} \times \frac{d}{d\tilde{s}} \tilde{F}_1 \right>
\]

Now

\[
\frac{d}{d\tilde{s}} \tilde{T} = \frac{d}{d\tilde{s}} N(s(\tilde{s})) = \left( -K(s) T(s) + \gamma(s) B(s) \right) \frac{d}{d\tilde{s}} s
\]
We know
\[ \hat{F}_1(s) = T(s), \]
so \[ \frac{d}{ds} \hat{T} \times F_1 \] is given by
\[ \left[ (-K(s) \frac{d}{ds} s) T(s) + (\gamma(s) \frac{d}{ds} s) B(s) \right] \times T = \]
\[ \gamma(s) \frac{d}{ds} s \ N \]
and our dot product is
\[ = \left< N, \gamma(s) \frac{d}{ds} s \ N \right> = \gamma(s) \frac{d}{ds} s = \frac{\gamma(s)}{K(s)}. \]

We now know that
\[ \hat{K}_1 = 1, \quad \hat{K}_2 = \frac{\gamma(s)}{K(s)}, \]
so
\[ \hat{K} = \sqrt{K_1^2 + K_2^2} = \sqrt{1 + \frac{\gamma^2}{K^2}} = \frac{\sqrt{\gamma^2 + K^2}}{K}. \]
It's a little weird to see the $k$ in the denominator, but it makes sense when you write

(total curvature of $\gamma = \int k(\gamma) d\gamma$

\[= \int \sqrt{k_1^2 + k_2^2} \, ds = \int \frac{\sqrt{k^2 + y^2}}{x} \, ds\]

\[= \int \sqrt{k^2 + y^2} \frac{d\gamma}{ds} \, ds = \int \sqrt{k^2 + y^2} \, ds\]

This is also recognizable as

$\int \left|N'(s)\right| ds = \text{length of normal indicatrix}$.

We can now see qualitatively the relationship between $T$ and curve geometry.
Further, we can see:

Proposition. Any two curves with the same tangent indicatrix have the same total curvature $\int x\,ds$ and same $\sqrt{x'^2+y'^2}\,ds$.

Here we mean "related by a reparametrization" when we say "same tangent indicatrix". Further, at corresponding points, the ratio $y/x$ is preserved as well.

Example. Scaling the curve by $\lambda$ scales $x$ and $y$ by $\lambda$ and reparametrizes the tangent indicatrix by $\lambda$.

A constant factor of $\lambda$
We are now able to observe some global features of the tangent indicatrix.

Proposition. A spherical curve $\gamma$ is the tangent indicatrix of a closed curve $\iff \gamma$ crosses every plane through the center of the sphere.

Proof. ($\Rightarrow$) Suppose $\gamma = T(s)$ for some curve $\alpha(s)$. We know that if $\alpha$ has length $L$, $\Omega \equiv \alpha(L) - \alpha(0) = \int_0^L \alpha'(s) \, ds = 0 = \int_0^L T(s) \, ds$.

Thus $\gamma(s) = T(s)$ has center of mass at $\bar{O}$. Further, for any plane through $\bar{O}$ with normal vector $\hat{n}$, $0 = \langle \hat{n}, \int_0^L T(s) \, ds \rangle = \int_0^L \langle \hat{n}, T(s) \rangle \, ds$.
so at some point, $\langle \hat{n}, T(s^*) \rangle = 0$, and $T$ crosses the plane. ($\forall$, for $\Rightarrow$)

This lets us classify:

1. not $T(s)$ for a closed curve.
2. $T(s)$ for some closed curve.

Then we have to prove ($\Leftarrow$).

The **convex hull** of a set is the intersection of all the halfspaces containing it.
It's a useful theorem that

\[ \text{conv}(S) = \{ p \mid p = \int_S d\mu \} \]

where \( d\mu \geq 0 \) everywhere on \( S \). This is true in & frightening generality (\( d\mu \) can contain point masses, or be even weirder!), but we'll only need to know

\[ \text{conv}(S) = \{ p \mid p = \int_S \omega(s) \otimes y(s) \, ds \} \]

for some weight function \( \omega(s) \geq 0 \), where \( s \) is arc length along the spherical curve \( y(s) \).

We now show that \( \tilde{o} \) is in \( \text{conv}(y) \). Given a halfspace \( h \) containing \( y \) with normal \( \tilde{n} \), slide in the \( \tilde{n} \) direction until we contact.
with the boundary plane. All subsequent planes cut $\gamma$ until we lose contact (forever) at the top of $\gamma$.

Since the plane through normal $\hat{n}$ through $\hat{o}$ does cut $\gamma$ (by hypothesis), if our halfspace $\hat{h}$ contains $\gamma$, it contains $\hat{o}$.

Thus $\exists$ some $\omega(s)$ so $\int y(s) \omega(s) ds = \hat{o}$

Reparametrize by $s^*$ so that $ds^* = \omega(s) ds$, and the curve

$$\alpha(s^*) = \int_0^{s^*} y(s^*) ds^*$$

has $\alpha'(s^*) = y(s^*)$ and $\int_0^L \alpha'(s^*) ds^* = 0$, so $\alpha(s^*)$ is a closed curve with tangent $\gamma$, as desired. $\square$