The Bishop Frame.

We now define another version of curve framing.

Suppose we start with some (arbitrary) smooth framing $F$ of $\gamma$, and consider the vector field

$$V(s) = \cos \theta(s) F_2(s) + \sin \theta F_3(s)$$

Then

$$V'(s) = -\sin \theta(s) \cdot \theta'(s) F_2(s)$$
$$+ \cos \theta(s) (-\alpha_{12}(s) F_1(s) + \alpha_{23}(s) F_3(s))$$
$$+ \cos \theta(s) \cdot \theta'(s) F_3(s)$$
$$+ \sin \theta (-\alpha_{23}(s) F_1(s) - \alpha_{23} F_2(s))$$

Gathering terms,

$$V'(s) = (-\alpha_{12}(s) \cos \theta(s) - \alpha_{13}(s) \sin \theta(s)) F_1(s)$$
$$(-\alpha_{23}(s) \sin \theta(s) - \sin \theta(s) \theta'(s)) F_2(s)$$
$$(+ \alpha_{23}(s) \cos \theta(s) + \cos \theta(s) \theta'(s)) F_3(s)$$
Observe that the $F_2, F_3$ coefficients are

$$-\sin \Theta(s) \left( \alpha_{23}(s) + \Theta'(s) \right)$$

and

$$\cos \Theta(s) \left( \alpha_{23}(s) + \Theta'(s) \right)$$

This means that if we set $\Theta'(s) = -\alpha_{23}(s)$ and integrate, we can define a family of frames such that:

$$F_2(s) = V_2(s) \text{ has } V'(s) = \kappa(s) T(s)$$

$F_2(s)$ depends on the initial angle $\Theta(0) = \Theta_0$, but any two frames with initial difference in angle $\Delta \Theta_0$ maintains this angular difference for all $s$.

We call this construction the Bishop frame, or relatively parallel adapted frame (RPAF).
The picture is

Traditionally, we write the structure equations

\[ T' = k_1 F_a + k_2 F_3 \]
\[ F_a' = -k_1 F_a \]
\[ F_3' = -k_2 F_3 \]

and call \( \alpha_{12}(s) = k_1(s) \) and \( \alpha_{13}(s) = k_2(s) \). These two functions are like \( x, y \):

\[ |T'| = x(s) = \sqrt{k_1(s)^2 + k_2(s)^2} \]

so \( x(s) \) is like radius in the \( k_1, k_2 \) plane.
To compute torsion, observe

\[ N(s) = \frac{T'(s)}{\|T'(s)\|} = \left( \frac{K_4}{K} \right) F_2 + \left( \frac{K_2}{K} \right) F_3 \]

\[ = \cos \theta \ F_2 + \sin \theta \ F_3 \]

Then

\[ N'(s) \cdot B(s) = \gamma(s) \text{     this is } B \]

\[ = \left( \Theta'(s) \right)(-\sin \theta \ F_2 + \cos \theta \ F_3) \]

\[ + \text{ something in } T \text{ direction; } B > \]

\[ = \Theta'(s). \]

So

\[ \gamma(s) = \Theta'(s), \text{    or } \Theta(s) = \int \gamma(s) \, ds \]

Since \( K_4 = \cos \theta, \ K_2 = \sin \theta, \) we see

\[ \frac{1}{\sqrt{K_2^2 + K_2^2}} \]

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\[ \int \gamma(s) \, ds \text{ is the polar } \Theta \text{ in the } K_2, K_2 \text{ plane.} \]
We can now see to what extent $k_1, k_2$ or $k_1, k_2$ determine the geometry of the curve $\gamma$.

We already know:

$\gamma = 0 \iff \gamma$ is planar (and has nonvanishing $k_3$).

In our new language, $\gamma = 0 \iff (k_1, k_2)$ is on a line through the origin.

Proposition (Bishop, 1990's).

$\gamma$ lies on a sphere $\iff (k_1, k_2)$ lies on a line not through the origin.

Proof.

$(\Rightarrow)$ Wlog, we may assume the sphere is centered at the origin and $\langle \gamma(s), \gamma(s) \rangle = r^2$. Differentiating, we see $\langle \gamma, \gamma' \rangle = 0$. 
Here's a neat trick. Choose any Bishop frame \((T, F_2, F_3)\) on \(y\). For any \(s\), this is a basis for \(\mathbb{R}^3\), so we can write \(y(s)\) in this basis. In principle,

\[ y(s) = \lambda_1(s) T(s) + \lambda_2(s) F_2(s) + \lambda_3(s) F_3(s) \]

but we know \(\langle y(s), T(s) \rangle = 0\), so \(\lambda_1(s) = 0\). This leaves us with

\[ y(s) = \lambda_2 F_2 + \lambda_3 F_3 \]

so

\[ y^1 = \lambda_2^1 F_2 + \lambda_2^1 k_1 T + \lambda_3^1 F_3 + \lambda_3^1 k_2 T \]

or

\[ \frac{\dot{y}}{\dot{t}} = (\lambda_2 k_1 + \lambda_3 k_2 - 1) T + \lambda_2^1 F_2 + \lambda_3^1 F_3 \]

But this means \(\lambda_2^1 = 0\), \(\lambda_3^1 = 0\) so the \(\lambda_i\) are constants.
And then we have

\[ \lambda_2 K_1(s) + \lambda_3 K_2(s) = 1 \]

which is exactly the equation of a line not through the origin! \( \Box \)

\((\Leftarrow)\) Suppose \( \lambda_2 K_1(s) + \lambda_3 K_2(s) = 1 \) for some constants \( \lambda_2, \lambda_3 \). Consider the vector

\[ \gamma(s) - (\lambda_2 F_2(s) + \lambda_3 F_3(s)) = \alpha(s). \]

If we differentiate,

\[ \alpha'(s) = T(s) - \lambda_2 K_1(s)T(s) - \lambda_3 K_2(s)T(s) \]

\[ = (1 - \lambda_2 K_1(s) - \lambda_3 K_2(s))T(s) \]

\[ = 0. \]

So \( \alpha(s) \) is a constant vector, \( \hat{\alpha} \). We claim this is the center of the sphere.
To prove it,

\[
\frac{d}{ds} \langle \gamma(s) - \alpha, \gamma(s) - \alpha \rangle = \\
= 2 \langle T(s), \gamma(s) - \alpha \rangle \\
= 2 \langle T(s), \lambda_2 F_2(s) + \lambda_3 F_3(s) \rangle \\
= 0
\]

So \( \gamma \) is on a sphere centered at \( \alpha \). \( \square \)