The tractrix.

A mass is located at $(0, 1)$ and pulled by a linkage of fixed length 1 moving along the x-axis at speed 1.

We know

\[ x(t) = t + \cos \theta \]
\[ y(t) = \sin \theta \]

because of the length-1 constraint.
Less obviously, the linkage is tangent to the curve, so we know that we have a triangle

\[ \theta(t) \]

\[ (x'(t), y'(t)) \]

Since

\[ \tan \phi = -\frac{y'(t)}{x'(t)} \]

we recall that \( y'(t) \) is negative.

and \( \phi = \pi - \theta \), the supplementary angle formula for \( \tan \) tells us that

\[ \tan \theta = \frac{y'(t)}{x'(t)} = \frac{\cos \theta \theta'(t)}{1 - \sin \theta \theta'(t)} \]

We can solve this formula for \( \theta'(t) \).
\[ \tan \theta \left(1 - \sin^2 \frac{\theta}{2}\right) = \cos \theta \theta' \]

\[ \tan \theta - \tan \theta \sin \theta \theta' = \cos \theta \theta' \]

\[ \tan \theta = \left(\cos \theta + \frac{\sin^2 \theta}{\cos \theta}\right) \theta' \]

- multiplying through by \( \cos \).

\[ \sin \theta = \theta' \]

We can solve this by separation of variables: \( \sin \theta = \frac{d\theta}{dt} \), so

\[ \int \frac{1}{\sin \theta} \, d\theta = \int 1 \, dt \]

and or

\[ \int \csc \theta \, d\theta = -\ln(\csc \theta + \cot \theta) + C \]

= \( t \)

for some constant \( C \).
at $t=0$, we have $\Theta = \frac{\pi}{2}$, so

\[
\csc \frac{\pi}{2} = 1, \quad \cot \frac{\pi}{2} = \frac{0}{1} = 0
\]

and $-\ln \left( \csc \frac{\pi}{2} + \cot \frac{\pi}{2} \right) = -\ln 1 = 0$.

This means $c = 0$. So

\[
\ell = -\ln \left( \csc \Theta + \cot \Theta \right).
\]

We want to solve this for $\Theta$.

Now

\[
\csc \Theta + \cot \Theta = \frac{1 + \cos \Theta}{\sin \Theta}
\]

We know

\[
\cos^2 \frac{\Theta}{2} = \frac{1 + \cos \Theta}{2}
\]

\[
\sin \Theta = 2 \cos \frac{\Theta}{2} \sin \frac{\Theta}{2}
\]
\[
\frac{1 + \cos \theta}{\sin \theta} = \frac{2 \cos^2 \frac{\theta}{2}}{2 \cos \frac{\theta}{2} \sin \frac{\theta}{2}} = \cot \frac{\theta}{2}
\]

(who said trig was useless!?) and

\[ t = + \ln \tan \frac{\theta}{2} \]

we switched from \cot to \tan, killing the minus sign.

so the tractrix is parametrized by \(\theta\) if we substitute this back into

\[ x(t) = t + \cos \theta \]
\[ y(t) = \sin \theta \]
to get

\[ x(\theta) = \cos \theta + \ln \tan \frac{\theta}{2} \]
\[ y(\theta) = \sin \theta \]

Looking at start, end we see

\[ \frac{\pi}{2} \leq \theta < \pi \]

What about a \( t \) parametrization?

Well, exp-ing \( t = \ln \tan \frac{\theta}{2} \), we get

\[ e^t = \tan \frac{\theta}{2} \]

We now have to solve for \( \sin \theta \) and \( \cos \theta \) in terms of \( \tan \frac{\theta}{2} \).

This is a trig exercise very similar to what we've already done.
Recall

\[ \sin \Theta = 2 \sin \frac{\Theta}{2} \cos \frac{\Theta}{2} \]
\[ \cos \Theta = \frac{1 - \tan^2 \frac{\Theta}{2}}{1 + \tan^2 \frac{\Theta}{2}} \]

and that we can use the sin and cos in terms of tan formulas to write

\[ \sin \frac{\Theta}{2} = \frac{\tan \frac{\Theta}{2}}{\sqrt{1 + \tan^2 \frac{\Theta}{2}}} \]
\[ \cos \frac{\Theta}{2} = \frac{1}{\sqrt{1 + \tan^2 \frac{\Theta}{2}}} \]

so we have

\[ \sin \Theta = \frac{2 \tan \frac{\Theta}{2}}{1 + \tan^2 \frac{\Theta}{2}} \]

Now plugging in \( e^t = \tan \frac{\Theta}{2} \),
we get

\[ \sin \Theta = \frac{2e^t}{1+e^{2t}} = \frac{2}{e^{-t}+e^t} = \text{sech} \ t \]

and

\[ \cos \Theta = \frac{1-e^{2t}}{1+e^{2t}} = \frac{e^{-t} - e^t}{e^{-t} + e^t} = -\tanh t \]

so we can also parametrize the tractrix by

\[ (t - \tanh t, \text{sech} \ t), \ t \geq 0. \]