Chapter 3

Independent Events

The word *probability*, in its mathematical acceptation, has reference to the state of our knowledge of the circumstances under which an event may happen or fail.
—*Collected Logical Works, Volume 2: The Laws Of Thought* by George Boole (Walton and Maberly, 1854)

A mother has 2 different pregnancies, each producing a single baby. What is the chance both babies are girls? What is the chance the older child is a girl? What is the chance exactly one of the babies is a girl? What is the chance neither baby is a girl? Are these probabilities the same? Why or why not?

3.1 Introduction

We have an intuitive understanding of the word "independence": Two events A and B are independent if the occurrence of one of the events does not affect the probability of occurrence of the other event. This is exactly right, but we need the concept of conditional probabilities (to be covered in Chapter 4), to use this viewpoint. In the present chapter, we define events A and B as **independent** if the probability that A and B both occur equals the probability that A occurs times the probability that B occurs. We will also discuss the notion of independence among more than two events. Afterwards, we will give examples of dependent events, as well as a very general fact about sequences of independent attempts, in which we are waiting for the first "good" result to occur.

Definition 3.1. Independence
Events A and B are called **independent** if

$$P(A \cap B) = P(A)P(B).$$

Definition 3.2. Dependence
Events A and B are called **dependent** if they are not independent. In other words, A and B are dependent if

$$P(A \cap B) \neq P(A)P(B).$$

We give a multitude of examples to clarify the concept of independence.

Example 3.3. Consider the birth of two children from two separate pregnancies (in particular, we are not considering the birth of twins, in which one baby's sex might affect the other).

If A is the event that the first baby is a girl, and B is the event that the second baby is a girl, then $P(A) = 1/2$, and $P(B) = 1/2$, and $P(A \cap B) = 1/4$, so $P(A \cap B) = P(A)P(B)$. Thus, events A and B are independent. This matches our traditional understanding of the word "independent," because the sex of the first baby does not affect the sex of the second baby.

Let C denote the event that both children are girls. Then $P(A \cap C) = 1/4$ but $P(A)P(C) = 1/8$, so A and C are dependent (intuitively, if C happens, then A must happen).

A question immediately arises:

Remark 3.4. Is "independent" the same thing as "disjoint"? Answer: "No"!

E.g., consider the outcome "girl, girl" in Example 3.3, which is found in both events A and B, and thus in $A \cap B$ too. So A and B are independent but are not disjoint.

More generally, consider the picture in Figure 3.1, for two different situations.

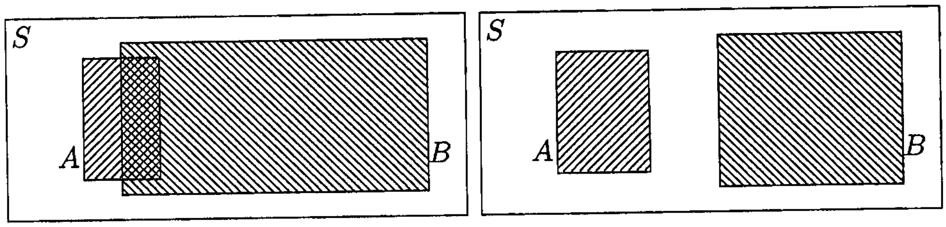

FIGURE 3.1: Left: Independent events A and B. Right: Disjoint events A and B.

Remark 3.5. Can A and B be both independent and disjoint? Only in a very special case: if $P(A) = 0$ or $P(B) = 0$.

Why? If A and B are disjoint, then they have no overlap, i.e., $A \cap B = \emptyset$. So $P(A \cap B) = P(\emptyset) = 0$. On the other hand, since A and B are independent, then $P(A)P(B) = P(A \cap B)$. So

$$P(A)P(B) = 0.$$

So either $P(A) = 0$ or $P(B) = 0$ (or both). In summary:

Remark 3.6. Independent vs Disjoint
If A and B both have positive probabilities, they cannot be both independent and disjoint.

Remark 3.7. When $P(A) = 0$, then A is independent from any event B, because, in such a case,

$$P(A \cap B) = 0 = P(A)P(B).$$

Example 3.8. When rolling a die, let A denote the event consisting of outcomes $\{1, 2, 3\}$, and let B denote the event consisting of outcomes $\{3, 4\}$, so $P(A) = 1/2$ and $P(B) = 1/3$. Also $A \cap B = \{3\}$, so $P(A \cap B) = 1/6$. So $P(A \cap B) = P(A)P(B)$, and this means that A and B are independent.

We will return to this scenario in Example 4.14.

Continuing the scenario, let $C = \{1, 2, 3, 5, 6\}$. So $P(C) = 5/6$. Also $P(B \cap C) = P(\{3\}) = 1/6$, but $P(B)P(C) = (1/3)(5/6) = 5/18 \neq 1/6$. So B, C are dependent.

Theorem 3.9. Subsets are dependent. If $A \subset B$ and neither $P(A) = 0$ nor $P(B) = 1$, then A, B are dependent.

To see this, consider such events A and B. We have $P(A \cap B) = P(A)$. Also $P(B) < 1$, so multiplying both sides by $P(A)$ (which is strictly positive) preserves the strict inequality. Thus $P(A)P(B) < P(A) = P(A \cap B)$, so A, B are dependent.

Theorem 3.10. Complements are dependent. If neither $P(A) = 0$ nor $P(A) = 1$, then A, A^c are dependent.

Note $P(A \cap A^c) = P(\emptyset) = 0$, but $P(A) \neq 0$ and $P(A^c) \neq 0$, so $P(A)P(A^c) \neq 0 = P(A \cap A^c)$, so A, A^c are dependent.

Example 3.11. Consider the songs from Exercise 2.1. Suppose that songs are chosen in such a way that each song is chosen at random, and repetitions are allowed, and every outcome is equally likely (an "outcome" is a particular song, not a genre).

1032	Alternative	83	Electronic	56	Metal
330	Blues	508	Folk	2718	Other
275	Books & Spoken	183	Gospel	1786	Pop
1468	Children's Music	82	Hip-Hop	403	R&B
921	Classical	564	Holiday	8286	Rock
6169	Country	537	Jazz	1432	Soundtrack
178	Easy Listening	106	Latin	216	World

Let A be the event that the first song is either blues or jazz. Let B be the event that the second song is jazz. Let C be the event that the third song is blues or rock.

Notice A and B are independent. Also, A and C are independent. Also, B and C are independent. In the scenario when song repetitions are allowed, the type of one song does not affect the types of other songs.

So far, we have only characterized what it means for a pair of events to be "independent." Now, we see the need to discuss the conditions that are necessary for a collection of three or more events to be collectively called "independent."

Definition 3.12. Independence of three events
A collection three events A, B, C is called (mutually) *independent* if all four of the following are satisfied:

$$
\begin{aligned}
P(A \cap B) &= P(A)P(B) \\
P(A \cap C) &= P(A)P(C) \\
P(B \cap C) &= P(B)P(C) \\
P(A \cap B \cap C) &= P(A)P(B)P(C)
\end{aligned}
$$

Example 3.11 (continued)
For instance, in the example above, about various combinations of blues and jazz and rock songs, the collection of events A, B, C is independent.

$$
\begin{aligned}
P(A \cap B) &= \left(\frac{330 + 537}{27{,}333}\right)\left(\frac{537}{27{,}333}\right) = P(A)P(B) \\
P(A \cap C) &= \left(\frac{330 + 537}{27{,}333}\right)\left(\frac{330 + 8286}{27{,}333}\right) = P(A)P(C)
\end{aligned}
$$

$$P(B \cap C) = \left(\frac{537}{27{,}333}\right)\left(\frac{330 + 8286}{27{,}333}\right) = P(B)P(C)$$

$$P(A \cap B \cap C) = \left(\frac{330 + 537}{27{,}333}\right)\left(\frac{537}{27{,}333}\right)\left(\frac{330 + 8286}{27{,}333}\right) = P(A)P(B)P(C)$$

Definition 3.13. Independence for a finite collection of events
A finite collection of events $A_1, A_2, \ldots, A_n$ are called (mutually) *independent* if, for *every subcollection* of the events, the probability of the intersection is equal to the product of the probabilities of the individual events in the collection.

Independence for an infinite collection of events
An infinite collection of events $A_1, A_2, \ldots$ is called (mutually) *independent* if every finite collection of the events is independent.

Example 3.14. Consider a student who flips twenty coins in a row. Let A_j denote the event that the jth coin shows a head. Then the events $A_1, \ldots, A_{20}$ are independent.

Example 3.15. Consider a student who flips coins for an arbitrarily long amount of time. As before, let A_j denote the event that the jth coin shows a head. Again, the individual coin flips do not impact each other, so the collection of all of the A_j's is independent.

Example 3.16. If A_j represents the event that there are two or more errors on the jth page of a book, then the collection of A_j's is perhaps independent, because the errors on the individual pages of a book should not affect the errors that occur on other pages of the book.

Example 3.17. The lifetimes of 100 randomly selected light bulbs are measured. Let A_j denote the event that the jth bulb lasts for at least 60 days. Then the collection of 100 events, $A_1, \ldots, A_{100}$, is independent.

Example 3.18. Two hundred customers' purchases are randomly inspected at the grocery store. The event A_j denotes the event that the jth customer purchased at least 3 dairy items. Then the collection of two hundred events, $A_1, \ldots, A_{200}$, are again (perhaps) independent.

When a finite or infinite sequence of random phenomenon are observed, and the results are recorded, we often use the word **trials** to denote such a collection of events. Trials are usually independent from each other, although they do not need to be independent. For instance, if we repeatedly draw cards from a deck, looking for the ace of spades, we might call each draw a "trial," regardless of whether the cards are replaced before the next trial. If the cards are replaced (and reshuffled!), then the trials are independent. If the cards are not replaced, then the trials are dependent.

In Examples 3.14 and 3.15, each coin flip is a trial. In Example 3.16, each examination of a page of a book is a trial. In Example 3.17, each test of the lifetime of a bulb is a trial. In Example 3.18, the examination of a customer's purchase is a trial.

Example 3.19. A student flips a coin until the tenth head appears. See Figure 3.2. Let A denote the event that at least 3 flips are needed between the 7th and 8th heads; let B denote the event that at least 3 flips are needed between the 8th and 9th heads. Then A and B are independent. The coin flips are trials.

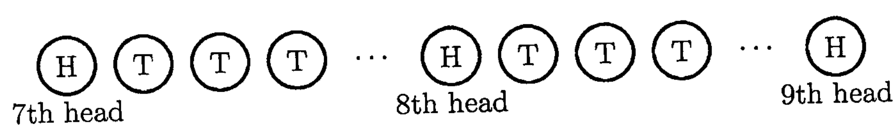

FIGURE 3.2: The number of flips between the 7th and 8th heads do not affect the number of flips between the 8th and 9th heads.

3.2 Some Nice Facts about Independence

Theorem 3.20. Independence among complements
When events are independent, their complements are too, i.e., if A, B are independent, then A^c, B are independent, A, B^c are independent, and A^c, B^c are independent too.

To see this, if A, B are independent, then

$$P(A)P(B) = P(A \cap B).$$

Subtracting both sides from $P(B)$ yields

$$P(B) - P(A)P(B) = P(B) - P(A \cap B),$$

i.e., $P(B)(1 - P(A)) = P(B \setminus A)$. So

$$P(B)P(A^c) = P(B \cap A^c),$$

i.e., A^c, B are independent.

Switching A and B, it follows too that if A, B are independent then A, B^c are independent too. Replacing B by B^c in the previous paragraph, we see that if A, B^c are independent then A^c, B^c are independent too.

Remark 3.21. These nice rules about independence between complements of events also extend to collections of three or more events.

Example 3.22. Two randomly chosen people are selected from a large college campus, and their heights are measured.

Let A denote the event that the height of the first person is 70 inches or greater; let B denote the event that the height of the second person is less than 68.5 inches. Then A and B are independent.

Let C denote the event that the first student's height is less than 68.5 inches. Then A and C are disjoint, so by Remark 3.5, A and C are dependent too.

3.3 Probability of Good Occurring before Bad

Example 3.23. Consider a very large container of carbonated beverages, of which 37% are apple flavor, 20% are orange flavor, and the other 43% are other flavors (cherry, lemon, etc.). If we repeatedly reach for a beverage until we get an apple or orange flavor, what is the probability that an apple flavored drink appears first?

Let A_n denote the event that the nth trial is apple and none of the earlier trials are apple or orange. Then we are looking for $P\big(\bigcup_{n=1}^{\infty} A_n\big)$.

Notice that the A_n's are disjoint. If A_3 occurs, then apple first appears on the 3rd trial, so apple cannot appear for the first time on the 1st trial, or 2nd trial, so neither A_1 nor A_2 can occur. Since the A_n's are disjoint, then

$$P\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} P(A_n)$$

Also $P(A_n) = (1 - 0.37 - 0.20)^{n-1}(0.37)$, since we need $n-1$ choices that are not apple or orange, followed by a choice that is apple.

So the desired probability is

$$\begin{aligned}\sum_{n=1}^{\infty} P(A_n) &= \sum_{n=1}^{\infty}(1 - 0.37 - 0.20)^{n-1}(0.37)\\ &= \frac{0.37}{1-(1-0.37-0.20)}\\ &= \frac{0.37}{0.37+0.20}\\ &= 0.65.\end{aligned}$$

This idea works much more generally and can be helpful in many situations in probability theory.

Theorem 3.24. Consider a sequence of independent trials, each of which can be classified as good, bad, or neutral, which happen (on any given trial) with probabilities p, q, and $1-p-q$, respectively. (We do not necessarily have $q = 1-p$ here, although that case is allowed.) Then the probability that something good happens before something bad happens is $p/(p+q)$.

To see this, consider a sequence of *independent* events $A_1, A_2, A_3, \ldots$, where the nth event indicates that something good happens on the nth trial, and neither good nor bad things happen on the previous trials. The same kind of reasoning from the apples and oranges argument works here.

We compute $P\big(\bigcup_{n=1}^{\infty} A_n\big)$. The A_n's are disjoint, since something good cannot happen *for the first time* on two different trials! Since the A_n's are disjoint, this gives

$$P\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} P(A_n).$$

Also, A_n occurs if $n-1$ neutral trials are followed by a good one, so

$$P(A_n) = (1-p-q)^{n-1}p.$$

So the desired probability is

$$P\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty}(1-p-q)^{n-1}p = \frac{1}{1-(1-p-q)}p = \frac{p}{p+q}.$$

So the probability something good happens before something bad happens is $p/(p+q)$.

3.4 Exercises

3.4.1 Practice

Exercise 3.1. Graduation. Jack and Jill are independently struggling to pass their last (one) class required for graduation. Jack needs to pass Calculus III, but he only has probability 0.30 of passing. Jill needs to pass Advanced Pharmaceuticals, but she only has probability 0.46 of passing. They work independently. What is the probability that at least one of them gets a diploma?

Exercise 3.2. Japanese pan noodles. Ten students order noodles at a certain local restaurant. Their orders are placed independently. Each student is known to prefer Japanese pan noodles 40% of the time (it is a very popular and tasty dish!).

a. What is the probability that all ten of the students order Japanese pan noodles?

b. What is the probability that none of the students order Japanese pan noodles?

c. What is the probability that at least one of the students orders Japanese pan noodles?

Exercise 3.3. Off to the races. Suppose Mike places three separate bets on three separate horse races at three separate tracks. Each bet is for a specific horse to win. His horse in race 1 wins with probability 1/5. His horse in race 2 wins with probability 2/5. His horse in race 3 wins with probability 3/5. What is the probability that he made the correct bet in exactly one of these three races?

Exercise 3.4. Early class. Consider these 3 independent trials: On Monday you wake up 45 minutes before class, and the probability that you get to class on time is 0.98. On Tuesday you wake up 32 minutes before class, and your chance of being on time is 0.71. On Wednesday you wake up very, very late, and your probability of being on time is only 0.16.

a. What is the probability that you were on time to class all 3 days?

b. What is the probability that you were never on time?

c. What is the probability that you were on time at least 1 day?

Exercise 3.5. Home for the holidays. A holiday flight from New York to Indianapolis has a probability of 0.75 each time it flies (independently) of taking less than 4 hours.

a. What is the probability that at least one of 3 flights arrives in less than 4 hours?

b. What is the probability that exactly 2 of the 3 flights arrive in less than 4 hours?

3.4.2 Extensions

Exercise 3.6. Hoops. Your sister is playing basketball. She makes 4 tosses to a lowered basketball hoop, and whether the ball goes in each time is independent of the other trials. Her chance of making the basket on a trial is 60%.

For each j with $0 \leq j \leq 4$, what is the probability that she makes exactly j baskets?

Exercise 3.7. Abstract art. A painter has three different jars of paint colors available, in colors green, yellow, and purple. She wants to paint something abstract, so she blindfolds herself, randomly dips her brush, and paints on the canvas. She continues trying paint jars, without replacement, until all three have been used. (Her assistant helps with this blindfolded process!) So sample space S is

$$S = \{(G, P, Y), (G, Y, P), (P, G, Y), (P, Y, G), (Y, G, P), (Y, P, G)\}.$$

Let A be the event that purple is found in the second jar tested by the painter. Let B be the event that green is found before yellow. Are events A and B independent?

Exercise 3.8. Even versus four or less. Roll a die. Let A be the event that the outcome on the die is an even number. Let B be the event that the outcome on the die is 4 or smaller. Let C be the event that the outcome on the die is 3 or larger.

a. Are A and B independent?

b. Are B and C independent?

Exercise 3.9. Vegetarian dilemma. In a very large collection of sandwiches, 40% are cheese, 45% have steak, and 15% have tofu. A person is vegetarian and therefore samples the sandwiches randomly until finding a cheese or tofu sandwich. What is the probability that they find a cheese sandwich before finding a tofu sandwich?

Exercise 3.10. Guessing on an exam. While taking a probability exam, you come to three questions that you have no clue how to answer. You would have known the answers if you had taken the time to study the night before instead of going to a party, but you did not make a good life choice, and you vow to never party on a school night again if you fail this exam. Each question on the exam is multiple choice with the correct answer being either a, b, c, d, or e. (Your guesses are independent.)

What is the probability that:

a. you randomly guess the right answer to all three questions?

b. you randomly guess the right answer to none of the three questions?

c. you randomly guess the right answer to exactly one of the three questions?

d. you randomly guess the right answer to exactly two of the three questions?

e. Do the probabilities in parts a–d sum to 1?

3.4.3 Advanced

Exercise 3.11. Can the sum be greater than 1? Is it possible to have two *independent* events A and B with the property that

$$P(A) + P(B) > 1\ ?$$

If your answer is "no," give a brief justification of why this is impossible.

If your answer is "yes," please give a brief example, in which you list the numbers $P(A)$ and $P(B)$ and also $P(A \cup B)$ and $P(A \cap B)$.

Exercise 3.12. Seating arrangements. Alice, Bob, Catherine, Doug, and Edna are randomly assigned seats at a circular table in a perfectly circular room. Assume that rotations of the table do not matter, so there are exactly 24 possible outcomes in the sample space.

Bob and Catherine are married. Doug and Edna are married.

Let T denote the event that Bob and Catherine are sitting next to each other. Let U be the event that Alice and Bob are sitting next to each other. Are events T and U independent?

Exercise 3.13. Political survey. On a large campus, 53% of the students are Democrats, and 47% are Republicans. A political survey is conducted. Assume that the students respond independently. How many students are needed, so that the probability of at least 1 Democratic participant exceeds 99%?