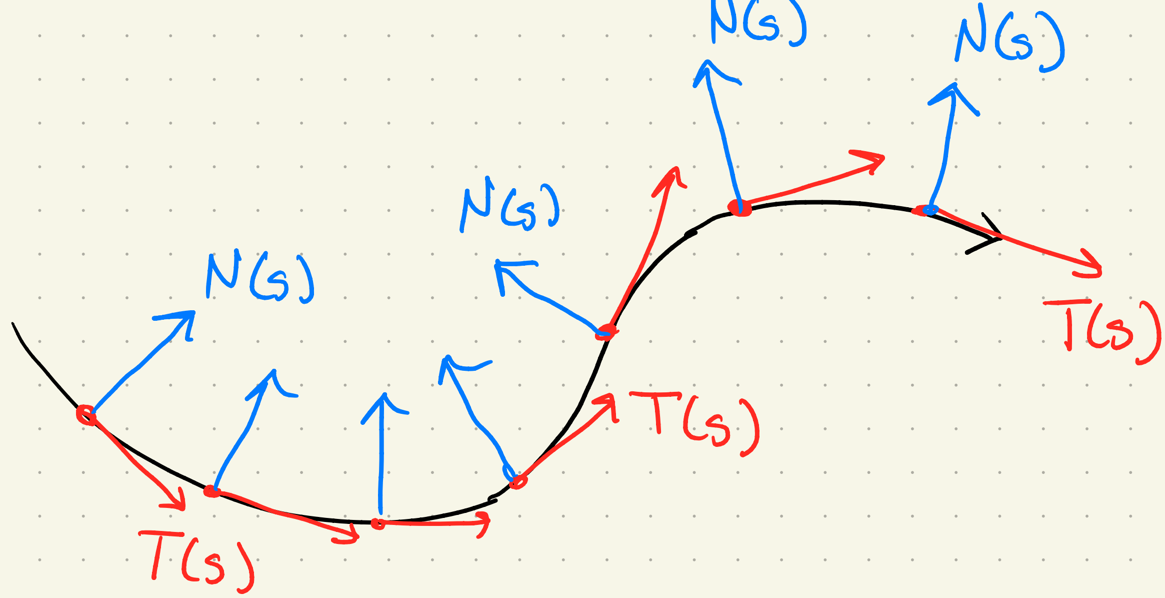
Curvature of Plane Curves

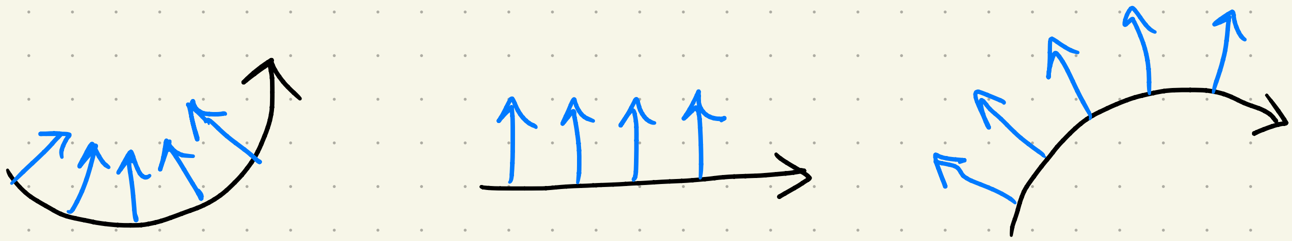
Suppose we have a curve $\vec{\alpha}: \mathbb{R} \to \mathbb{R}^2$, parametrized by arclength.

<u>Definition</u>. If $A: \mathbb{R}^2 \to \mathbb{R}^2$ is the linear map "rotate by $+\pi/2$" given by $A\vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$, we let $\vec{x}^{\perp} = A\vec{x}$.

<u>Definition</u>. The unit tangent vector $T(s) = \vec{\alpha}'(s)$. The unit normal vector $N(s) = T(s)^{\perp}$.



<u>Definition</u>. The Gauss curvature of $\vec{\alpha}(s)$ is defined by $K_{\pm}(s) := \langle \vec{\alpha}''(s), N(s) \rangle$.

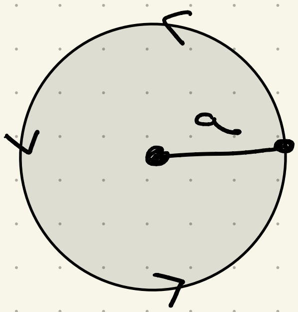


positive curvature

zero curvature

negative curvature

<u>Example</u>. Circle of radius a, ccw parametrization.



$$\vec{\alpha}(s) = \left(a\cos\left(\tfrac{s}{a}\right), a\sin\left(\tfrac{s}{a}\right)\right).$$

$$\vec{\alpha}'(s) = \left(-\sin\left(\tfrac{s}{a}\right), \cos\left(\tfrac{s}{a}\right)\right).$$

$$\vec{\alpha}''(s) = \left(-\tfrac{1}{a}\cos\left(\tfrac{s}{a}\right), -\tfrac{1}{a}\sin\left(\tfrac{s}{a}\right)\right).$$

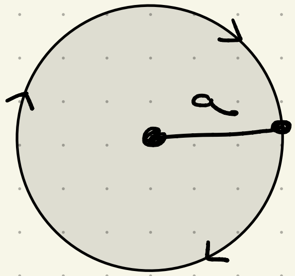
$$T(s) = \vec{\alpha}'(s) = \left(-\sin\left(\tfrac{s}{a}\right), \cos\left(\tfrac{s}{a}\right)\right).$$

$$N(s) = T(s)^{\perp} = \left(-\cos\left(\tfrac{s}{a}\right), -\sin\left(\tfrac{s}{a}\right)\right)$$

$$K_{\pm}(s) = \langle \vec{\alpha}''(s), N(s) \rangle = +\frac{1}{a}\cos^2\frac{s}{a} + \frac{1}{a}\sin^2\frac{s}{a}$$

$$= \frac{1}{a}.$$

Example. Circle of radius a, cw parametrization.



$$\vec{\alpha}(s) = \left(a\cos\left(-\frac{s}{a}\right), a\sin\left(-\frac{s}{a}\right)\right).$$

$$\vec{\alpha}'(s) = \left(+\sin\left(-\frac{s}{a}\right), -\cos\left(-\frac{s}{a}\right)\right).$$

$$\vec{\alpha}''(s) = \left(-\frac{1}{a}\cos\left(-\frac{s}{a}\right), -\frac{1}{a}\sin\left(-\frac{s}{a}\right)\right).$$

$$T(s) = \vec{\alpha}'(s) = \left(+\sin\left(-\frac{s}{a}\right), -\cos\left(-\frac{s}{a}\right)\right)$$

$$N(s) = T(s)^{\perp} = \left(+\cos\left(-\frac{s}{a}\right), +\sin\left(-\frac{s}{a}\right)\right)$$

$$K_{\pm}(s) = \langle \vec{\alpha}''(s), N(s) \rangle = -\frac{1}{a}\cos^2\left(-\frac{s}{a}\right) - \frac{1}{a}\sin^2\left(-\frac{s}{a}\right)$$
$$= -\frac{1}{a}.$$

Proposition. If $\vec{\beta}(s) = \vec{\alpha}(-s)$, $K_{\pm}^{\beta}(s) = -K_{\pm}^{\alpha}(s)$.

Proof. We just compute:

$$\vec{\beta}'(s) = -\vec{\alpha}'(-s), \quad \vec{\beta}''(s) = +\vec{\alpha}''(-s),$$

$$T^{\beta}(s) = -T^{\alpha}(s), \quad N^{\beta}(s) = -N^{\alpha}(s),$$

$$K_{\pm}^{\beta}(s) = \langle \vec{\beta}''(s), N^{\beta}(s) \rangle$$
$$= \langle \vec{\alpha}''(s), -N^{\alpha}(s) \rangle = -K_{\pm}^{\alpha}(s). \quad \boxed{/\!/}$$

<u>Proposition</u>. If $\vec{\alpha}(t)$ is a regular parametrization, then

$$\kappa_{\pm}(t) = \frac{\langle \vec{\alpha}''(t), \vec{\alpha}'(t)^{\perp} \rangle}{\langle \vec{\alpha}'(t), \vec{\alpha}'(t) \rangle^{3/2}}$$

<u>Proof</u>. We know there exists some function $t(s)$ so that $\vec{\alpha}(t(s))$ is arclength parametrized. Further,

$$t'(s) = \frac{1}{s'(t)} = \frac{1}{\|\vec{\alpha}'(t)\|}.$$

Now

$$\frac{d}{ds}\vec{\alpha}(t(s)) = \vec{\alpha}'(t(s)) \cdot t'(s) = T(t(s))$$

so

$$N(t(s)) = T(t(s))^{\perp} = \vec{\alpha}'(t(s))^{\perp} \cdot t'(s)$$

$\perp$ is a linear map, so the scalar $t'(s)$ comes out

$$\frac{d^2}{ds^2}\vec{\alpha}(t(s)) = \vec{\alpha}''(t(s)) \cdot t'(s)^2 + \vec{\alpha}'(t(s)) \cdot t''(s)$$

product rule for scalar multiplication

Now that we're done differentiating, we'll stop writing "$t(s)$" and just write "t".

$$K_{\pm}(t) = \left\langle \frac{d^2}{ds^2}\vec{\alpha}(t),\, N(t) \right\rangle$$

$$= \left\langle \vec{\alpha}''(t)\cdot(t')^2 + \vec{\alpha}'(t)\,t'',\; \vec{\alpha}'(t)^{\perp}\cdot t' \right\rangle$$

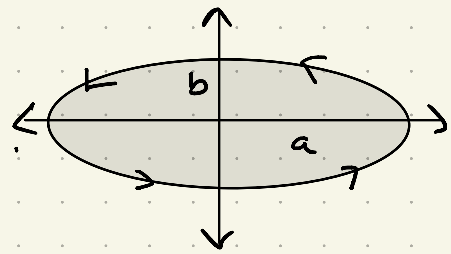
these are orthogonal!

$$= \langle \vec{\alpha}''(t), \vec{\alpha}'(t)^{\perp} \rangle \cdot (t')^3$$

$t' = \dfrac{1}{\|\vec{\alpha}'(t)\|}$

$$= \frac{\langle \vec{\alpha}''(t), \vec{\alpha}'(t)^{\perp} \rangle}{\langle \vec{\alpha}'(t), \vec{\alpha}'(t) \rangle^{3/2}}. \qquad \blacksquare$$

Example. Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



$$\vec{\alpha}(t) = (a\cos t, b\sin t)$$

$$\vec{\alpha}'(t) = (-a\sin t, b\cos t)$$

$$\vec{\alpha}'(t)^{\perp} = (-b\cos t, -a\sin t)$$

$$\vec{\alpha}''(t) = (-a\cos t, -b\sin t)$$

$$K_{\pm}(t) = \frac{\langle \vec{\alpha}''(t), \vec{\alpha}'(t)^{\perp} \rangle}{\langle \vec{\alpha}'(t), \vec{\alpha}'(t) \rangle^{3/2}} = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}$$

so $K_{\pm}(0) = K_{\pm}(\pi) = \dfrac{ab}{b^3} = \dfrac{a}{b^2}$

$K_{\pm}(\pi/2) = K_{\pm}\left(\dfrac{3\pi}{2}\right) = \dfrac{ab}{a^3} = \dfrac{b}{a^2}$

Note: if $b=a$, this reduces to our previous calculation for circle.

<u>Lemma.</u> If $\vec{v}(t), \vec{\omega}(t)$ are unit vectors in $\mathbb{R}^2$ with $\langle \vec{v}(t), \vec{\omega}(t) \rangle = 0$ for all t,

$$\langle \vec{v}', \vec{\omega} \rangle = -\langle \vec{v}, \vec{\omega}' \rangle$$

$$\langle \vec{v}', \vec{v} \rangle = 0$$

$$\langle \vec{\omega}', \vec{\omega} \rangle = 0.$$

<u>Proof.</u> All of these come from the product rule. $\langle \vec{v}(t), \vec{\omega}(t) \rangle = 0$ for all t, so

$$0 = \frac{d}{dt} \langle \vec{v}(t), \vec{\omega}(t) \rangle$$

$$= \langle \vec{v}'(t), \vec{\omega}(t) \rangle + \langle \vec{v}(t), \vec{\omega}'(t) \rangle$$

Further, because $\vec{v}(t)$ is a unit vector for all t, $\langle \vec{v}(t), \vec{v}(t) \rangle = 1$ for all t,

$$0 = \frac{d}{dt} \langle \vec{v}(t), \vec{v}(t) \rangle$$

$$= 2 \langle \vec{v}'(t), \vec{v}(t) \rangle.$$

The proof that $\langle \vec{\omega}', \vec{\omega} \rangle = 0$ is similar. ▨

<u>Definition</u>. The <u>orthogonal</u> group

$$O(n) = \{A \in Mat_{n\times n} \mid A^T = A^{-1}\}$$

and the <u>special</u> <u>orthogonal</u> <u>group</u>

$$SO(n) = \{A \in O(n) \mid \det A = 1\}.$$

We think of $O(n)$ as "orthonormal bases" and $SO(n)$ as "positively oriented orthonormal bases".

<u>Proposition</u>. If $\vec{\alpha}(t)$ is a regular plane curve, $F(t) = \begin{bmatrix} \uparrow & \uparrow \\ T(t) & N(t) \\ \downarrow & \downarrow \end{bmatrix} \in SO(2)$.

<u>Proof</u>. $T(t)$ is a unit vector, so

$T(t) = (\cos\theta(t), \sin\theta(t))$ for some $\theta(t)$.

$N(t) = T(t)^{\perp} = (-\sin\theta(t), \cos\theta(t))$. So

$$F^T F = \begin{bmatrix} \leftarrow T(t) \rightarrow \\ \leftarrow N(t) \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow \\ T(t) & N(t) \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

and $\det F = \cos^2\theta(t) + \sin^2\theta(t) = 1$. ▨

__Definition.__ If $\vec{\alpha}(s)$ is an arclength parametrized curve in $\mathbb{R}^2$, we call the map $F(s) = \begin{bmatrix} \uparrow & \uparrow \\ T(s) & N(s) \\ \downarrow & \downarrow \end{bmatrix}$ the __Frenet frame__, and note $F : \mathbb{R} \to SO(2)$.

__Proposition.__ In this case,

$$T'(s) = \kappa_{\pm}(s) N(s)$$

$$N'(s) = -\kappa_{\pm}(s) T(s)$$

(we call these the __Frenet equations.__)

__Proof.__ T, N is an orthonormal basis, so

$$T' = \underbrace{\langle T', T \rangle}_{=0 \text{ (lemma)}} T + \underbrace{\langle T', N \rangle}_{=\kappa_{\pm} \text{ (defn)}} N = \kappa_{\pm} N$$

$$N' = \underbrace{\langle N', T \rangle}_{\substack{= -\langle N, T' \rangle \text{ (lemma)} \\ = -\kappa_{\pm} \text{ (defn)}}} T + \underbrace{\langle N', N \rangle}_{=0 \text{ (lemma).}} N = -\kappa_{\pm} T.$$

$\blacksquare$

Note that the Frenet equations imply

$$F^T F' = \begin{bmatrix} \leftarrow T \rightarrow \\ \leftarrow N \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow \\ T' & N' \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 0 & -\kappa_t \\ \kappa_t & 0 \end{bmatrix}$$

<u>Proposition</u>. If $F : \mathbb{R} \to O(n)$, then $F' = FS$ where $S^T = -S$ (S is skew-symmetric).

<u>Proof</u>. Set $S = F^T F'$. Then

$$FS = \underbrace{F F^T}_{= I \text{ b/c } F \text{ is in } SO(n)} F' = F'.$$

Now $FF^T = I$ for all t so

$$0 = \frac{d}{dt} F F^T = F' F^T + F(F^T)' = S + F(F^T)'.$$

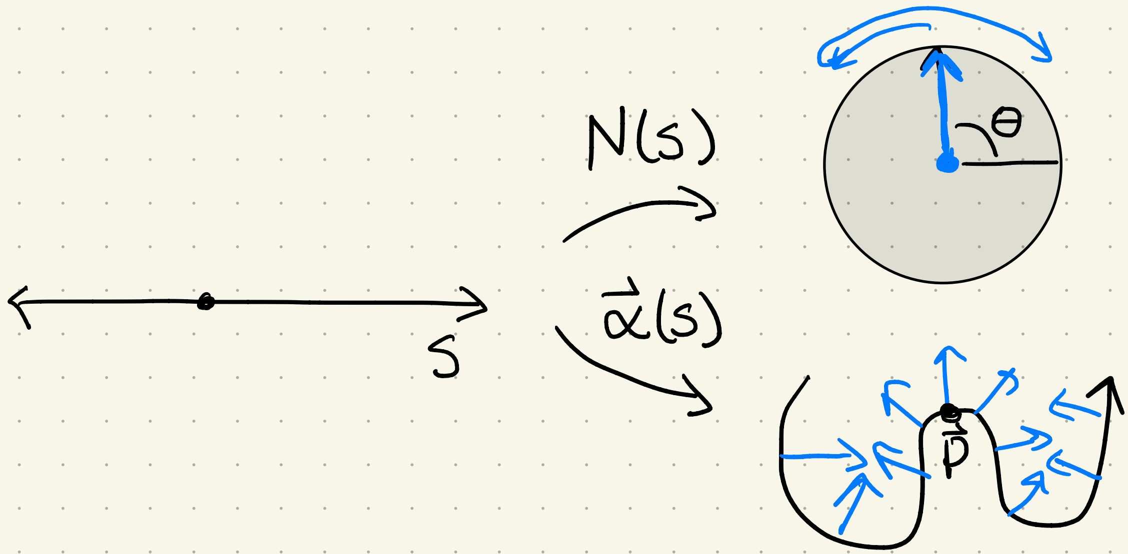
Since F^T has the same coordinate functions as F (just written in different places in the grid), $(F^T)' = (F')^T$. Thus

$$F(F^T)' = F(F')^T = (F' F^T)^T = S^T.$$

We now have $S + S^T = 0$.

$\blacksquare$

But what *is* Gauss curvature?



Assume $\vec{\alpha}(s)$ is parametrized by arclength, and consider $N(s)$ sweeping back and forth around the unit circle, parametrized ccw by θ.

$$\frac{d\theta}{ds} = \langle N', N^{\perp} \rangle$$

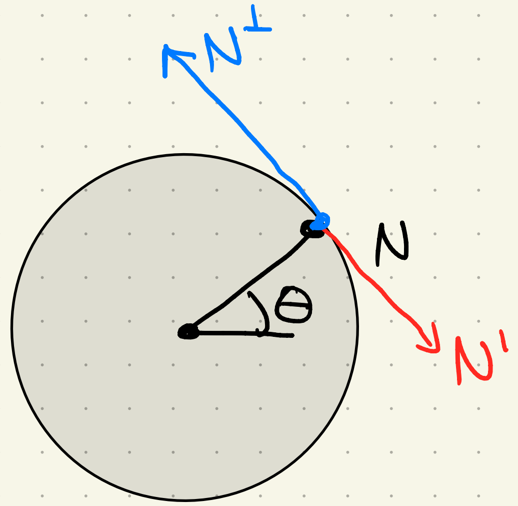
$$= \langle -\kappa_{\pm} T, (T^{\perp})^{\perp} \rangle$$

Frenet equations   defn of N

$$= \kappa_{\pm} \langle -T, -T \rangle$$

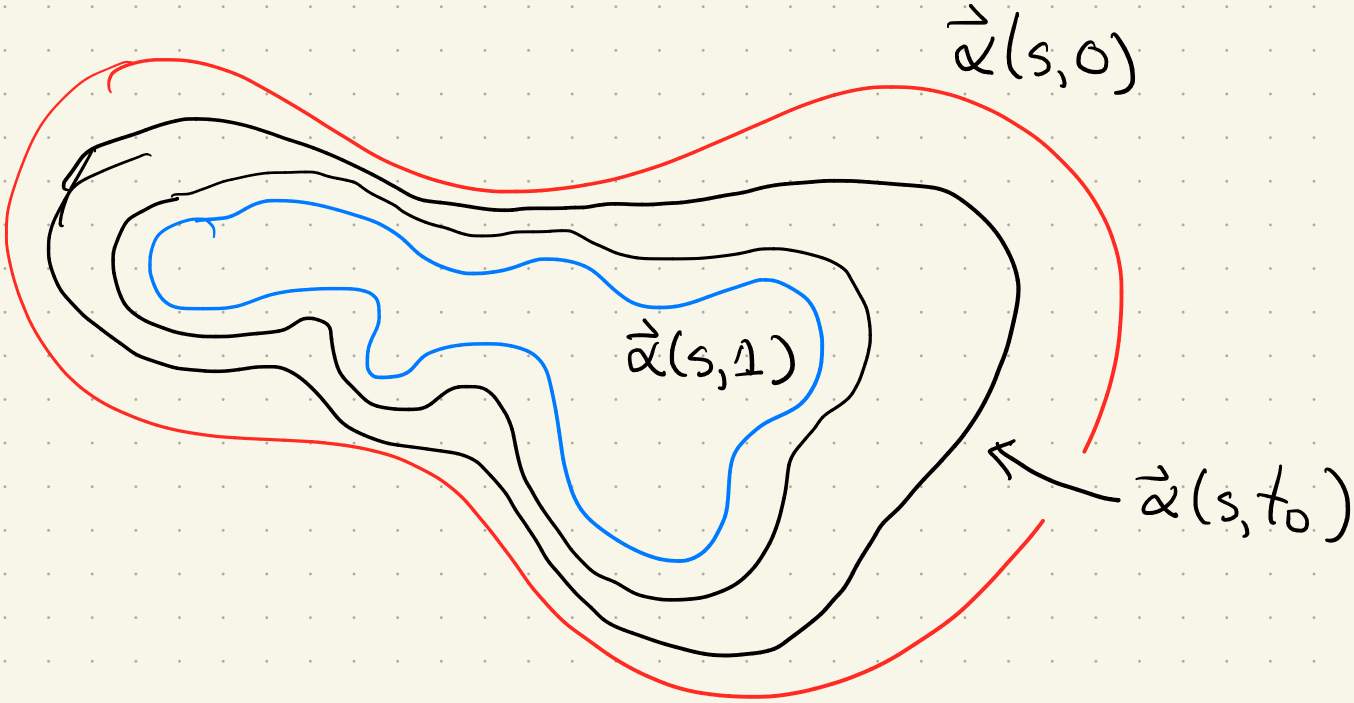
⊥ rotates by π/2, so ⊥⊥ rotates by π

$$= \kappa_{\pm}.$$



Definition. If $\vec{\alpha}$ is a regular closed curve in $\mathbb{R}^2$ with length L, then we call $\int_0^L \kappa_\pm(s)\,ds$ the total curvature.

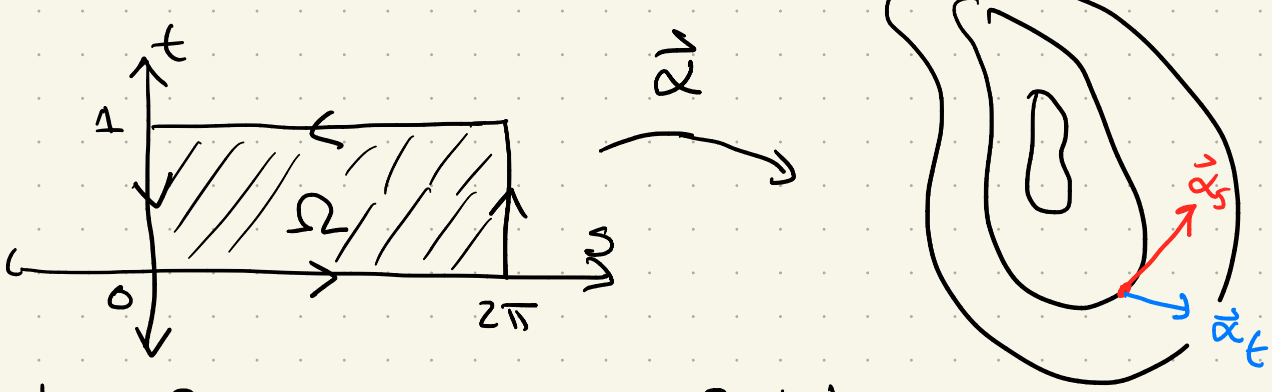
Definition. A regular homotopy is a map $\vec{\alpha}: S^1 \times [0,1] \to \mathbb{R}^2$ so that for each fixed t_0, $\vec{\alpha}(s,t_0)$ is a regular parametrization of a curve in $\mathbb{R}^2$.
We say $\vec{\alpha}$ joins $\vec{\alpha}(s,0)$ and $\vec{\alpha}(s,1)$.



Theorem. If $\vec{\alpha}^0$ and $\vec{\alpha}^1$ are regular closed curves joined by a regular homotopy, they have the same total curvature.

Proof. Since total curvature doesn't change under scaling or reparametrization, we may assume all $\vec{\alpha}(s,t_0)$ have length 2π and $\vec{\alpha}: S^1 \times [0,1] \to \mathbb{R}^2$ has s as an arclength parameter on each curve.

Consider



and define a vector field

$$\vec{V} = \left(\langle N_s, N^\perp \rangle, \langle N_t, N^\perp \rangle \right)$$

Now we recall <u>Green's</u> <u>theorem</u>,

$$\int_{\partial\Omega} \vec{V}\, d\text{length} = \iint_{\Omega} \nabla \times \vec{V}\, d\text{Area}$$

where if $\vec{V} = (f, g)$, $\nabla \times \vec{V} = g_s - f_t$.

We compute

$$\nabla \times \vec{V} = \langle N_t, N^\perp \rangle_s - \langle N_s, N^\perp \rangle_t$$

$$= \cancel{\langle N_{ts}, N^\perp \rangle} + \langle N_t, N_s^\perp \rangle$$

$$- \cancel{\langle N_{st}, N^\perp \rangle} - \langle N_t, N_s^\perp \rangle$$

mixed partials commute

Now $\langle N, N \rangle \equiv 1$, so $\langle N_s, N \rangle = 0$ and $\langle N_t, N \rangle = 0$. But $N, N^\perp$ are an orthonormal basis for $\mathbb{R}^2$, so this implies N_s and N_t are colinear – both are multiples of $N^\perp$.

Thus $\langle N_s, N_t^\perp \rangle = \langle N_t, N_s^\perp \rangle = 0$.

By Green's theorem, we now have

$$0 = \int_{\partial\Omega} \vec{V}\, d\text{length}$$

$$= \int_0^{2\pi} \langle \vec{V}(s,0),(1,0)\rangle\, ds + \cancel{\int_0^1 \langle \vec{V}(2\pi,t),(0,1)\rangle dt}$$

$$- \int_0^{2\pi} \langle \vec{V}(s,1),(1,0)\rangle\, ds - \cancel{\int_0^1 \langle \vec{V}(0,t),(0,1)\rangle dt}$$

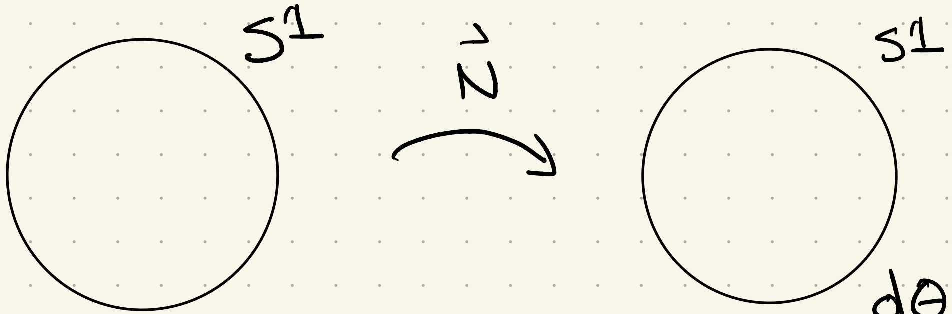
the curves are closed, so $\vec{V}(0,t) = \vec{V}(2\pi,t)$

$$= \int_0^{2\pi} K_{\pm}^{0}(s)\, ds - \int_0^{2\pi} K_{\pm}(1)\, ds,$$

which completes the proof! ∎

If you took 3500/3510:

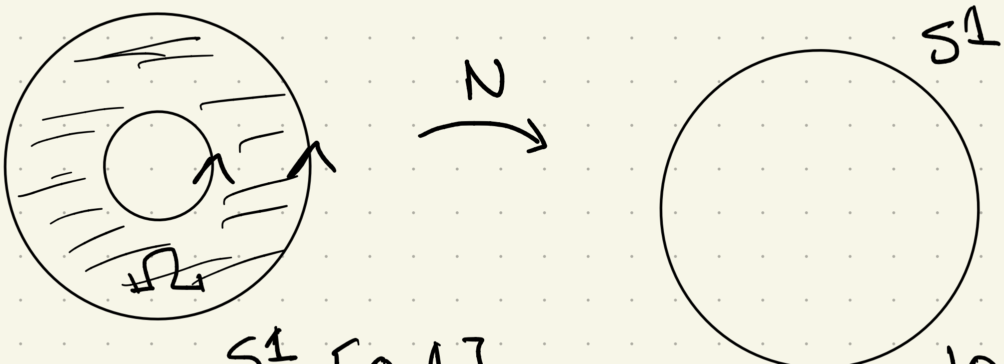
For a single curve, we have



$d\theta$ = length 1-form

Total curvature $= \int N^* d\theta$.

For a homotopy between curves



$S^1 \times [0,1]$

$d\theta$ = length 1-form

defn of boundary

$$\int_{S^1 \times \{1\}} N^* d\theta - \int_{S^1 \times \{0\}} N^* d\theta = \int_{\partial(S^1 \times [0,1])} N^* d\theta = \int_{S^1 \times [0,1]} d(N^* d\theta)$$

Stokes theorem

no 2-forms on the 1-manifold S^1

$$= \int_{S^1 \times [0,1]} N^*(d(d\theta)) = \int_{S^1 \times [0,1]} N^*(0) = 0.$$

pullback and d commute

Whitney-Graustein Theorem.

Every regular closed plane curve has total curvature $2\pi n$, where $n \in \mathbb{Z}$ and is regularly homotopic to

1) an n-covered circle, if $n \neq 0$
2) the figure 8 curve, if $n = 0$.

Proof. Alas, beyond scope of this class!

Example.