Meshes and Manifolds

Definition. Suppose C is a collection of (finite) sets so that if $S \in C$, and $S' \subset S$, then $S' \in C$.

We call C an *abstract simplicial complex*, and if $S \in C$ has $K+1$ elements, we call S a *K-simplex*.

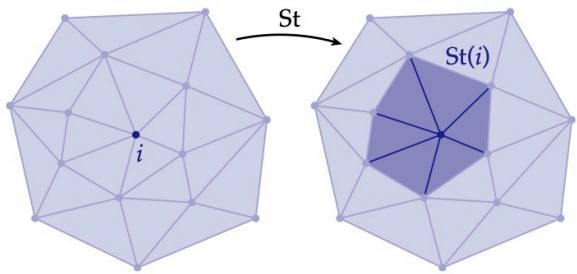
Example. $C = \{\{1,2\},\{1\},\{2\}\}$.

$$C = \{\{1,2,3\},\{2,3,4\},\{2,3\},\{1,3\},\{1,2\},\{2,4\},\{1\},\{2\},\{3\},\{4\}\}.$$

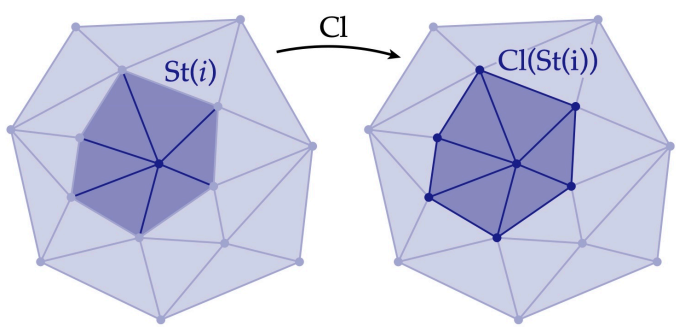
A *subcomplex* $K' \subset K$ is a subset which is also a simplicial complex.

A *pure-K* simplicial complex has every subset contained in a K-simplex.

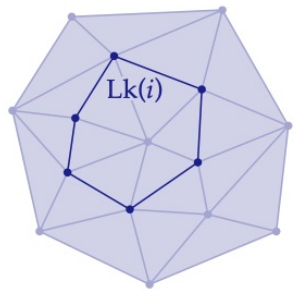
Definition. The **star** of a subset S is the collection of all simplices $\sigma \in K$ that contain any simplex in S.



The **closure** of a subset $S \subset K$ the smallest subcomplex of K containing S. For instance, if $S = St(i)$:



The link of a subset S of K is $Lk(S) = Cl(St(S)) - St(Cl(S))$.



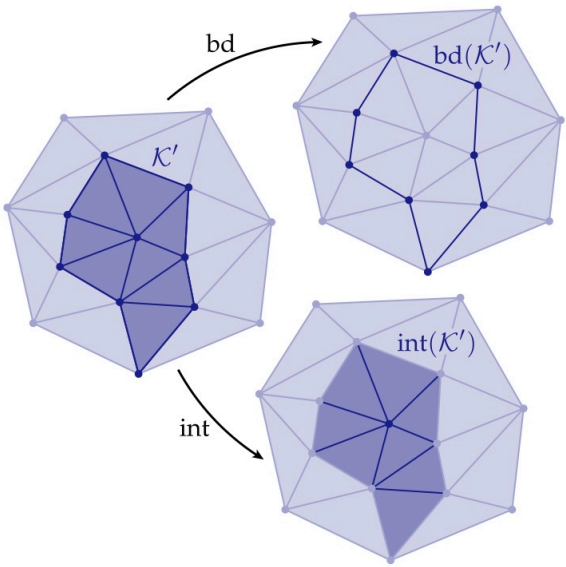
$\mathrm{Lk}(i) = \mathrm{Cl}(\mathrm{St}(i)) \setminus \mathrm{St}(\mathrm{Cl}(i))$

These are "disks" and "spheres" around vertices and boundaries of normal neighborhoods otherwise.

Similarly, if we have a subcomplex $K' \subset K$, then boundary$(K') = bd(K')$

$$bd(K') = \{\sigma \in K' \mid \sigma \text{ is a (strict) proper subset of exactly one simplex in } K'\}$$

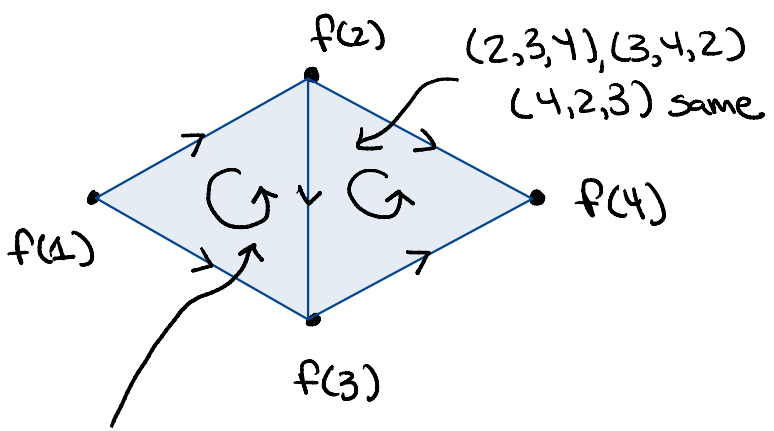
and the interior $\text{int}(K') = K' - \text{bd}(K')$.



Definition. An oriented abstract simplicial complex C gives each $S \in C$ an equivalence class of orderings related by even permutations, which we call positive (all other orderings are negative).

Example.

$$C = \{\{1,2,3\}, \{2,3,4\}, \{2,3\}, \{1,3\}, \{1,2\}, \{2,4\}, \{1\}, \{2\}, \{3\}, \{4\}\}.$$

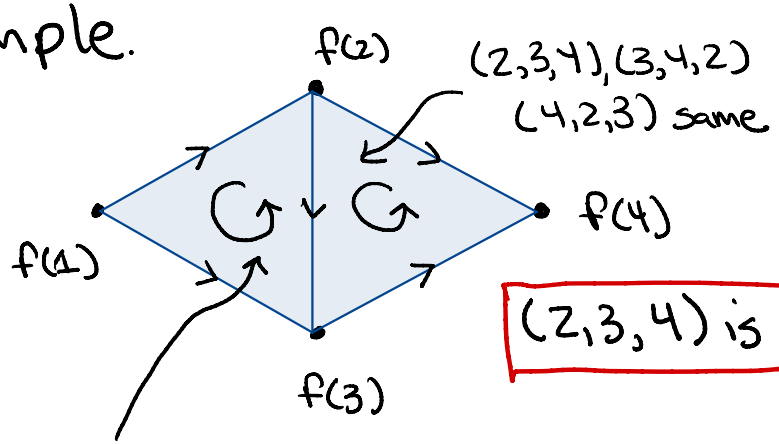


$(1,3,2), (3,2,1), (2,1,3)$ all the same.

Suppose two oriented K-simplices S and T have $\#S \cap T = K$. Then each has a unique element $s \notin T$ and $t \notin S$. Choose any order on the shared vertices $(v_0, \dots, v_{K-1})$. Then the orientations on S, T compatible if

$(v_0, \dots, v_{K-1}, s)$
$(v_0, \dots, v_{K-1}, t)$ opposite signs in S, T.

Example.



(2,3,4) is positive

(1,3,2), (3,2,1), (2,1,3) all the same.

(2,3,1) is negative

A simplicial complex has a consistent orientation if every pair of ℓ-simplices in X which intersect in an $\ell-1$ simplex have compatible orderings.

Definition. A pure-K simplicial complex is a pseudomanifold if every $(K-1)$-simplex is contained in 1 or 2 K-simplices.

(Meunier, Combinatorial Stokes formulae)

Definition. A simplicial pseudomanifold is a simplicial n-manifold if the link of every vertex is homeomorphic to S^{n-1}.

Checking this is:

- trivial for $n = 1, 2, 3$
- In NP ∩ co-NP by Schleimer (2011) and Lackeby (2021) but suspected to be in P for $n = 4$
- Completely open for $n = 5$
- Known to be uncomputable for $n > 5$. Novikov, 1968.

Plan. We are going to study geometry (and topology) of simplicial n-manifolds for relatively small n.

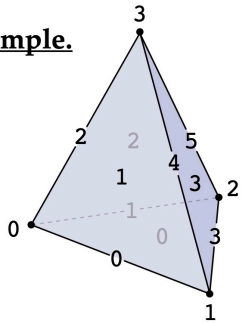
Definition. Let K be a simplicial complex with n_i i-simplices (for $i \in 0, \ldots, K$), and suppose for each i the simplices are numbered. The i-th incidence matrix

$$E^i \in Mat_{n_{i+1} \times n_i}(\mathbb{Z}) = \left[\; e^i_{pq} \; \right]$$

where

$$e^i_{pq} = \begin{cases} 1, & \text{if } i\text{-simplex } q \\ & \text{is a subset of} \\ & i+1\text{-simplex } p \\ 0, & \text{if not.} \end{cases}$$

Example.



$$E^0 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array} \qquad E^1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

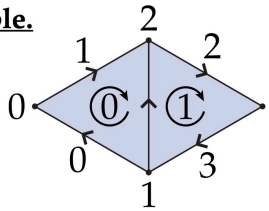
As mathematicians, we prefer

Definition. Let K be an oriented simplicial complex with n_i i-simplices for $i \in 0, \ldots, K$ and suppose for each i the simplices are oriented and also numbered. The ith signed incidence matrix

$$E^i \in Mat_{n_{i+1} \times n_i}(\mathbb{Z}) = \left[e^i_{pq} \right]$$

$$e^i_{pq} = \begin{cases} +1, & \text{if } i\text{-simplex } p \text{ is a subset of } i+1\text{-simplex } q \text{ and orientations agree} \\ -1, & \text{if } i\text{-simplex } p \text{ is a subset of } i+1\text{-simplex } q \text{ and orientations agree} \\ 0, & \text{otherwise} \end{cases}$$

Example.



$$E^0 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \end{array} \qquad E^1 = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

These matrices are coboundary operators.

Definition. $E_i = (E^i)^T \in Mat_{n_i \times n_{i+1}}$ is the boundary operator.

To justify the name, we define

Definition. An i-chain is a vector $\vec{v} \in \mathbb{Z}^{n_i}$ or $\mathbb{R}^{n_i}$. We think of it as a formal linear combination of i-simplices.

$E_i \vec{v}$ is the boundary of the i-chain.

Two facts are useful:

1) If $\vec{v}$ assigns 1 to each K-simplex incident to vertex j, then $E_{K-1} \vec{v}$ is the $(K-1)$-chain of $\text{Link}_K(j)$.

2) $E_{i-1} E_i = 0$ (a boundary has no boundary), or $E^i E^{i-1} = 0$ (d of d of a form is zero).

Now each row of E^1 contains only $(i+1)$ nonzero elements, but we often have millions of rows, so they are stored in sparse matrix format (ccs).

$$E^0 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \end{array}$$

$$E^0 = \begin{bmatrix} 0 & -1 \\ -0 & 2 \\ -2 & 3 \\ 1 & -3 \\ -1 & 2 \end{bmatrix}$$

$$E^1 = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$E^1 = \begin{bmatrix} 0 & 1 & -4 \\ 2 & 3 & 4 \end{bmatrix}$$

Suppose you want to list all vertices incident to a edge i.

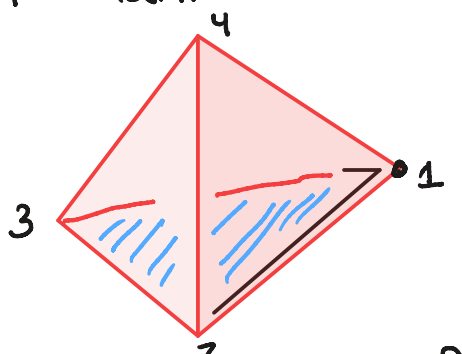
$O(1)$: just read the ith row of E^0.

Suppose you want to list all edges incident to vertex i.

$O(n)$: you have to search E^0 for $\pm i$

A different structure: combinatorial maps.

Definition. Suppose we have a simplicial K-manifold.



A dart represents a flag

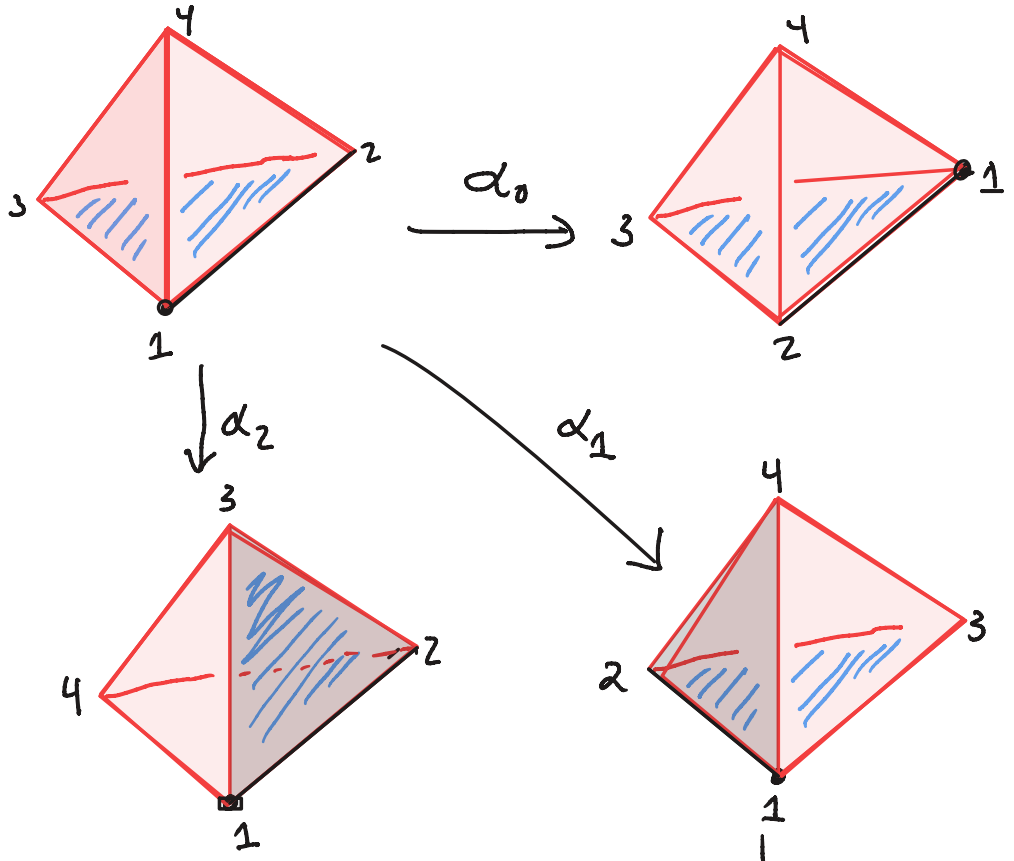
0-cell $\subset$ 1-cell $\subset$ 2-cell $\subset$ ... $\subset$ K-cell

and a flag induces an orientation on the K-cell, by ordering the vertices.

Darts are numbered, and there is a special constant called $\emptyset$ (empty).

We define $\alpha_i(\emptyset) = \emptyset$ and

α_i – change the i-dimensional simplex in the flag (only)



If there was another 3-simplex, α_3 would take us to the 3-simplex sharing the blue face with this one. As it is, $\alpha_3(d) = \emptyset$.

Lemma. Every α_i has $\alpha_i(d) = \emptyset$ or $\alpha_i(\alpha_i(d)) = d$ and reverses orientation of the K-simplex.

Proof. An orientation of a K-simplex is an equivalence class of vertex orderings. The α_i map changes this by a transposition.

Definition. The i-th pointer map $\beta_i = \alpha_i \circ \alpha_0$ is an orientation preserving map from darts to darts, with the exception that $\beta_0 = \alpha_0 \alpha_1$.

1) d is i-free if $\beta_i(d) = \emptyset$

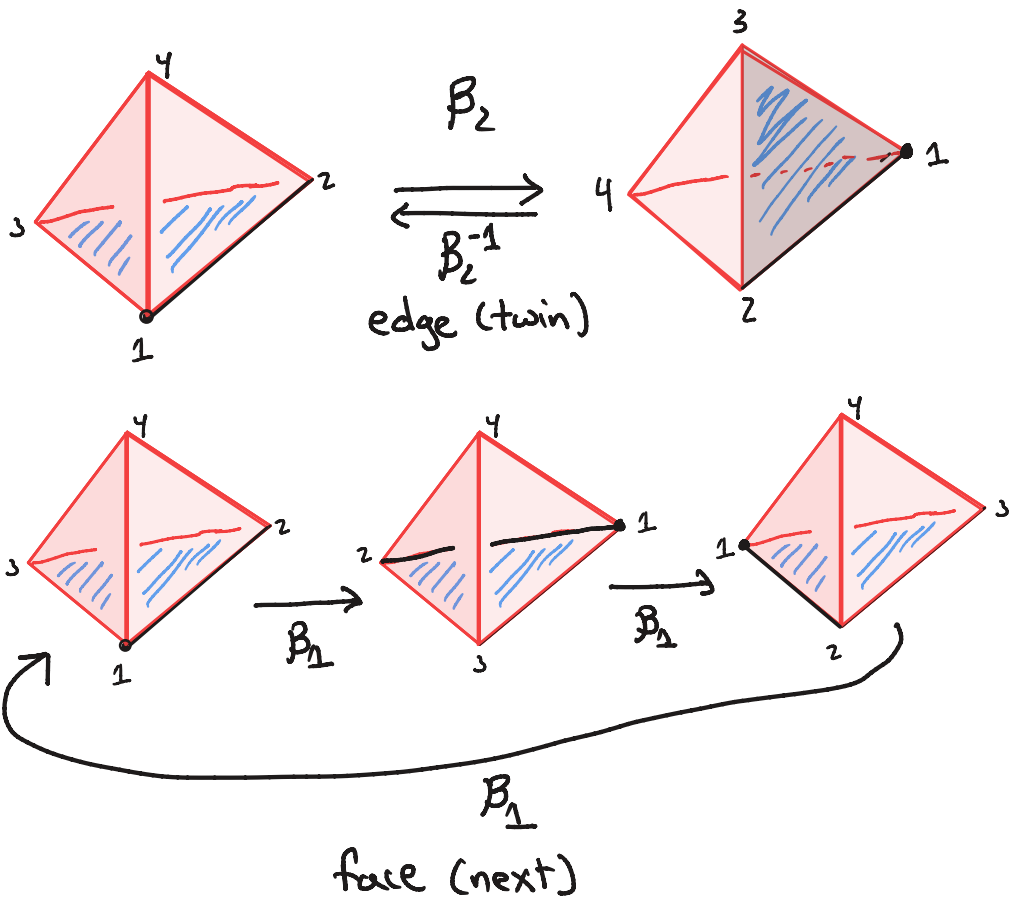
2) for $2 \leq i \leq K$ if d is not i-free, then $\beta_i(\beta_i(d)) = d$.

3) β_0 and β_1 are inverses.

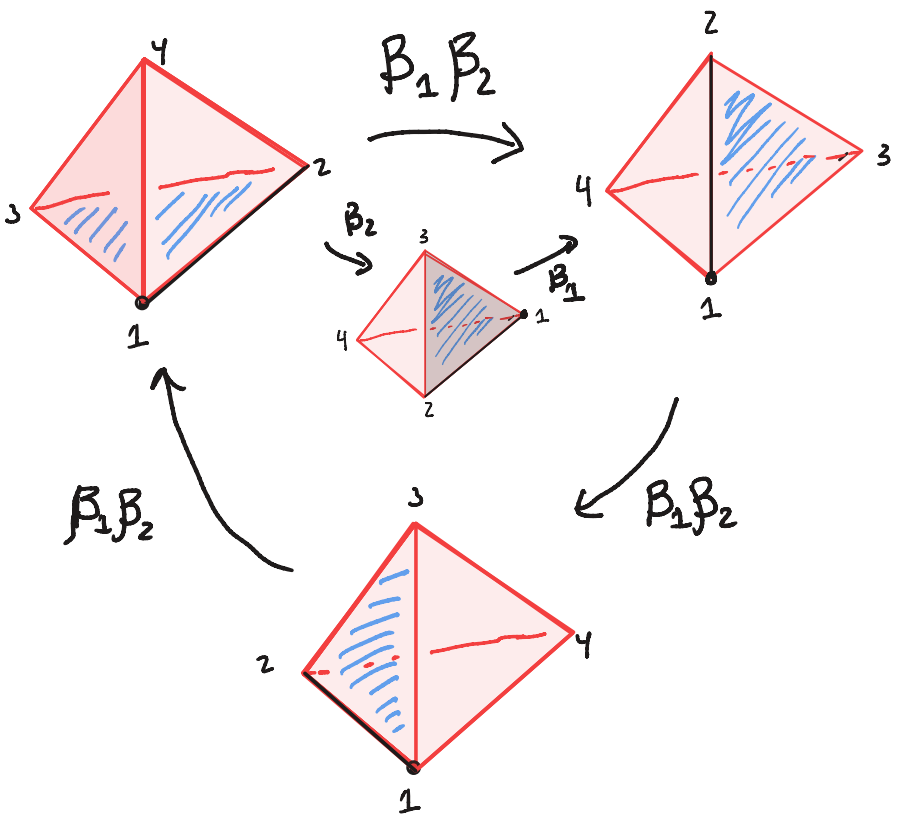
A combinatorial map is without i-boundary if there are no i-free darts and without boundary if it is without i-boundary for $i \in 0, \ldots, K$.

The cells of the complex correspond to orbits of darts:

i-cell of d $\longleftrightarrow$ orbit of d under $(i>0)$ $\beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots \beta_K$ and their inverses



0-cell of d $\longleftrightarrow$ orbit of d under $\beta_i\beta_j$ $(i<j)$ and their inverses.



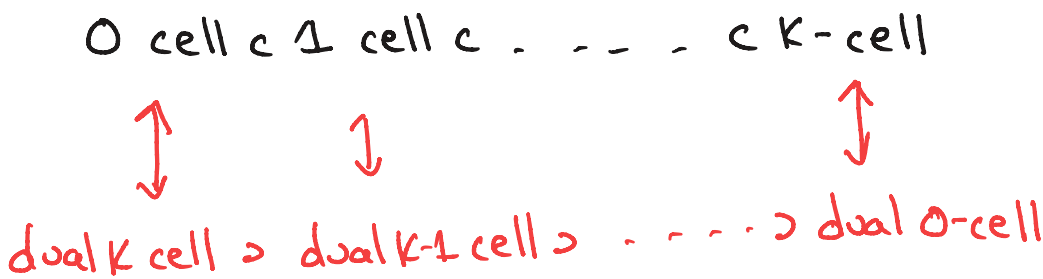
Proposition. Two cells are incident $\Leftrightarrow$ their dart orbits are not disjoint (they share any dart).

Note that cells of different dimensions may be incident.

Two i-cells c_1, c_2 are adjacent if there are darts $d_1 \in c_1$, $d_2 \in c_2$ so that $d_1, d_2 \in c_3$ for some $(i-1)$-cell c_3.

Proposition. If H is any even set, β_2 any fixed-point free involution, and β_1 any permutation, then (H, β_1, β_2) is a compact orientable surface with faces, edges, vertices given by orbits of β_1, β_2, and $\beta_1\beta_2$.

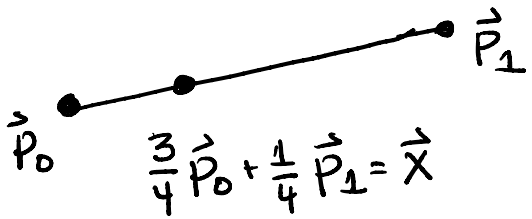
Fun fact: Poincaré duality on combinatorial maps is trivial - the darts of K and the dual are just read backwards



and the maps are the same - just reindexed!

<u>Definition</u>. A collection of points $\vec{p}_0, \ldots, \vec{p}_k \in \mathbb{R}^n$ are <u>affine independent</u> if $\{\vec{p}_1 - \vec{p}_0, \ldots, \vec{p}_k - \vec{p}_0\}$ are linearly independent.

Definition. Given $\vec{p}_0, \dots, \vec{p}_k \in \mathbb{R}^n$, if if $\vec{x} = t_0\vec{p}_0 + \dots + t_k\vec{p}_k$ we call the t_i the barycentric coordinates of $\vec{x}$.



Proposition. The t_i are unique $\Leftrightarrow$ the p_i are affine independent.

Definition. If $\vec{p}_0, \dots, \vec{p}_k$ are affine indep., the open geometric simplex Δ_k with vertices $\vec{p}_0, \dots, \vec{p}_k$ is

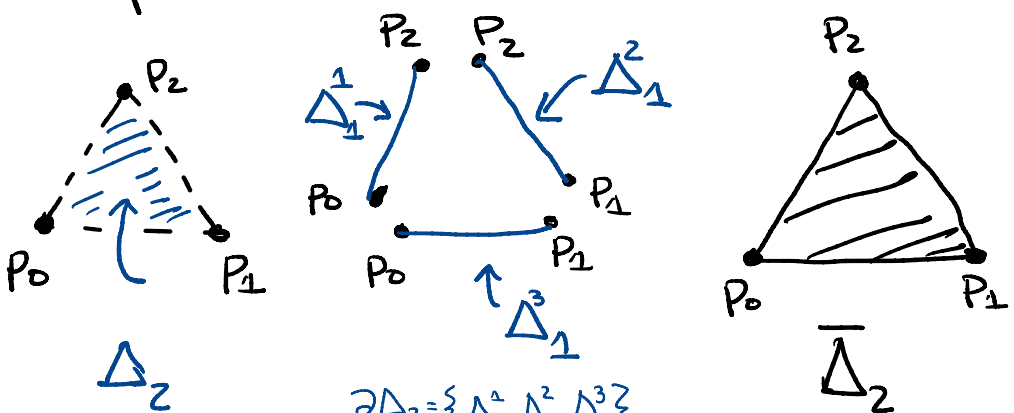
$$\left\{ \vec{x} \in \mathbb{R}^n \;\middle|\; \vec{x} = \sum t_i \vec{p}_i,\ t_i > 0,\ \sum t_i = 1 \right\}.$$

The boundary $\partial\Delta_k$ consists of all open geometric simplices Δ_e whose vertices $\{q_0,\dots,q_e\} \subsetneq \{p_0,\dots,p_k\}$.

The closed geometric simplex $\overline{\Delta}_k$

$$\left\{ \vec{x} \;\middle|\; \vec{x} = \sum t_i \vec{p}_i,\ t_i \geq 0,\ \sum t_i = 1 \right\}.$$

Example.



A geometric realization of an abstract simplicial complex C is a map $f: \bigcup_{S \in C} \to \mathbb{R}^n$, so that

1) For each $S = \{x_0, \dots, x_k\} \in C$, $f(x_0), \dots, f(x_k)$ are affine independent.

2) For each $S = \{x_0, \dots, x_k\}$, $T = \{y_0, \dots, y_e\} \in S$, the geometric simplices Δ_k and Δ_e with vertices $f(x_i)$ and $f(y_j)$ have $\overline{\Delta}_k \cap \overline{\Delta}_e = \overline{\Delta}_p$ where Δ_p is the geometric simplex with vertices $f(S \cap T)$.

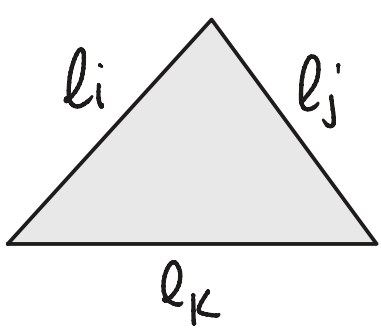
A geometric realization is an embedding of the PL manifold in $\mathbb{R}^n$.



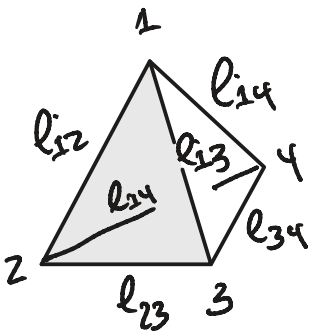
But what about <u>intrinsic</u> geometry?

Idea. An intrinsic geometry or metric on a simplicial n-manifold is an assignment of positive lengths to the 1-simplices so that there exists a geometric realization of each K-simplex with the given lengths.

Example. For a triangulated surface, lengths must obey triangle inequalities



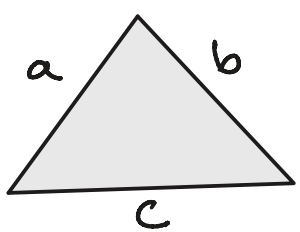
$$\ell_i + \ell_j \geq \ell_K$$
$$\ell_j + \ell_K \geq \ell_i$$
$$\ell_K + \ell_i \geq \ell_j$$



For a tetrahedron, any collection of three lengths must obey triangle inequalities

and any sum of 3 triangle areas must be larger than the fourth.

Now Heron's formula:



$$A^2 = -\frac{1}{16}\begin{vmatrix} 0 & a^2 & b^2 & 1 \\ a^2 & 0 & c^2 & 1 \\ b^2 & c^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

tells us that the tetrahedral inequalities are quadratic in edgelengths.

So you'd suspect that higher dimensional "simplex inequalities" involve higher degree polynomials.

"Gram embedding theorem": Given n^2 constants ℓ_{ij}^2, $i,j \in 1,\ldots,n$, they are the pairwise squared distances between n points in $\mathbb{R}^d \iff$

$$\ell_{ij}^2 \geq 0, \quad \ell_{ij}^2 = \ell_{ji}^2, \quad \ell_{ii}^2 = 0$$

and the $(n-1)\times(n-1)$ symmetric matrix

$$g_{ij} = \frac{1}{2}\left(\ell_{1i}^2 + \ell_{1j}^2 - \ell_{ij}^2\right)$$

is positive semidefinite and has rank d.

Proof. The symmetric matrix G has a spectral decomposition

$$G = U \Lambda U^T$$

where the entries of Λ are ≥ 0. By the rank hypothesis, at most d columns of Λ are nonzero, so taking the first d columns of U to be U_d, we also have

$$G = \underset{(n-1)\times d}{(U_d \Lambda_d^{1/2})} (U_d \Lambda_d^{1/2})^T. \quad \blacksquare$$

A bit weaker, one has

Schoenberg Embedding Theorem.

A symmetric matrix Q is a (squared) distance matrix $\Leftrightarrow$ the quadratic form $Q(\vec{v}) = \vec{v}^T Q \vec{v}$ is negative semidefinite on $\vec{1}^\perp$.

Here $\vec{1}^\perp$ is the orthogonal complement of the vector of all 1's.

Final!

Definition. A metric on an oriented simplicial K-manifold is a vector $\vec{g}$ in $(\mathbb{R}^{n_1})^+$ so that for every K-simplex the $(K+1)\times(K+1)$ matrix of entries in $\vec{g}$ corresponding to edges of the simplex obeys Schoenberg's criterion.

Lemma. Q satisfies Schoenberg $\Leftrightarrow$ $\left(I - \frac{1}{K+1}\vec{1}\vec{1}^T\right) Q \left(I - \frac{1}{K+1}\vec{1}\vec{1}^T\right)$ is negative semidefinite.

Proof. The matrices on either side of Q are the projectors to $\vec{1}^{\perp}$. ▨

For simplicial K-manifolds of reasonable size (millions of simplices, $K \leq 10$) this is extremely fast to check.

Application. Regge calculus.

Let K be a 4-dimensional oriented simplical manifold with a metric $\vec{g}$.

This manifold obeys the discrete version of the vacuum field equations (Regge equations) $\Leftrightarrow$
For each 1-simplex e_j, we have

$$0 = \frac{1}{2}\sqrt{g_j}\sum_{\substack{\text{2-simplices}\\ \Delta_K\\ \text{where } e_j\\ \text{is an edge}\\ \text{of } \Delta_K}} \epsilon_{\Delta_K} \cot \alpha_{e_j}^{\Delta_K}$$

where $\alpha_{e_j}^{\Delta_K}$ is the angle in triangle Δ_K across from edge e_j, and

$$\epsilon_{\Delta_K} = 2\pi - \sum_{\substack{\text{4-simplices} \\ \mathcal{Y}_\ell \text{ containing} \\ \Delta_K \text{ as a 2-simplex}}} \theta_{\Delta_K}^{\mathcal{Y}_\ell}$$

while

$\theta_{\Delta_K}^{\mathcal{Y}_\ell}$ = angle at which the two 3-simplices inside $\mathcal{Y}_\ell$ sharing Δ_K meet

"dihedral angle"

Solutions of the Regge equations on finer and finer meshes converge to continuum solutions of the vacuum field equations.

There are many interesting issues involved in solving these equations numerically... but at least you now can see that the equations don't involve anything all that exotic.