Gauss Bonnet and the meaning of K

We want to understand the Gauss curvature. To do that, we'll need to talk about area.

Recall:

<u>Definition</u>. Given vectors $\vec{a}, \vec{b} \in T_pM$, the oriented area of the parallelogram spanned by $\vec{a}, \vec{b}$ is

$$\sigma_M(\vec{a}, \vec{b}) = \langle \vec{a} \times \vec{b}, \vec{n} \rangle_{\mathbb{R}^3}.$$

We note that

$$\sigma_M(\vec{X}_u, \vec{X}_v) = \langle \vec{X}_u \times \vec{X}_v, \vec{n} \rangle_{\mathbb{R}^3}$$

$$= \frac{\langle \vec{X}_u \times \vec{X}_v, \vec{X}_u \times \vec{X}_v \rangle_{\mathbb{R}^3}}{|\vec{X}_u \times \vec{X}_v|}$$

$$= |\vec{X}_u \times \vec{X}_v|$$

Now recall that

$$\vec{a} \times (\vec{b} \times \vec{c}) = \langle \vec{a}, \vec{c} \rangle \vec{b} - \langle \vec{a}, \vec{b} \rangle \vec{c}$$

for any vectors. And further,

$$\langle \vec{a}, \vec{b} \times \vec{c} \rangle = \langle \vec{b}, \vec{c} \times \vec{a} \rangle = \langle \vec{c}, \vec{a} \times \vec{b} \rangle.$$

Thus we have

$$\begin{aligned}
\langle \vec{p} \times \vec{q}, \vec{r} \times \vec{s} \rangle &= \langle \vec{r}, \vec{s} \times (\vec{p} \times \vec{q}) \rangle \\
&= \langle \vec{r}, \langle \vec{s}, \vec{q} \rangle \vec{p} - \langle \vec{s}, \vec{p} \rangle \vec{q} \rangle \\
&= \langle \vec{s}, \vec{q} \rangle \langle \vec{r}, \vec{p} \rangle - \langle \vec{s}, \vec{p} \rangle \langle \vec{r}, \vec{q} \rangle
\end{aligned}$$

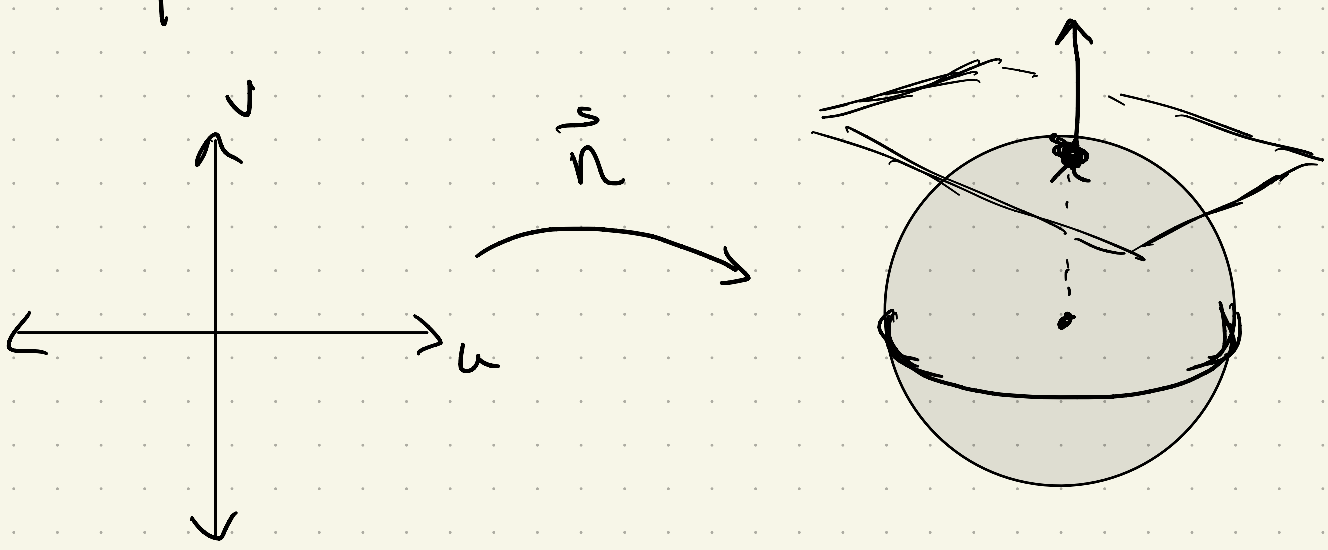
and in particular,

$$\begin{aligned}
\sigma_M(\vec{x}_u, \vec{x}_v) &= \sqrt{\langle \vec{x}_u \times \vec{x}_v, \vec{x}_u \times \vec{x}_v \rangle} \\
&= \sqrt{\langle \vec{x}_v, \vec{x}_v \rangle \langle \vec{x}_u, \vec{x}_u \rangle - \langle \vec{x}_u, \vec{x}_v \rangle^2} \\
&= \sqrt{EG - F^2}
\end{aligned}$$

Note: This is a particular example in differential forms, where we have proved that

$$\vec{X}^* \sigma_M = \sqrt{EG - F^2}\, du \wedge dv$$

Now let's do something interesting. Suppose we think of $\vec{n}(u,v)$ as a parametrization of the sphere:



The sphere has the interesting property that the normal vector is equal to the position vector.

So we have

$$\sigma_{S^2}(\vec{n}_u, \vec{n}_v) = \langle \vec{n}_u \times \vec{n}_v, \vec{n} \rangle$$

normal to sphere
and position $\frac{x_u \times x_v}{|x_u \times x_v|}$

$$= \frac{\langle \vec{n}_u \times \vec{n}_v, \vec{x}_u \times \vec{x}_v \rangle}{|\vec{x}_u \times \vec{x}_v|}$$

But we can do the side computation

$$\langle \vec{n}_u \times \vec{n}_v, \vec{x}_u \times \vec{x}_v \rangle = \langle \vec{x}_v, \vec{n}_v \rangle \langle \vec{x}_u, \vec{n}_u \rangle - \langle \vec{x}_u, \vec{n}_v \rangle \langle \vec{x}_v, \vec{n}_u \rangle$$

Recalling that $\langle \vec{n}, \vec{x}_v \rangle \equiv \langle \vec{n}, \vec{x}_u \rangle \equiv 0$,

$$= \langle \vec{x}_{vv}, \vec{n} \rangle \langle \vec{x}_{uu}, \vec{n} \rangle - \langle \vec{n}, \vec{x}_{uv} \rangle^2$$

$$= \ell n - m^2$$

which means that

$$\sigma_{S^2}(\vec{n}_u, \vec{n}_v) = \frac{lm - n^2}{\sqrt{EG - F^2}}$$

or $\vec{n}^* \sigma_{S^2} = \dfrac{lm - n^2}{\sqrt{EG-F^2}}\, du \wedge dv.$

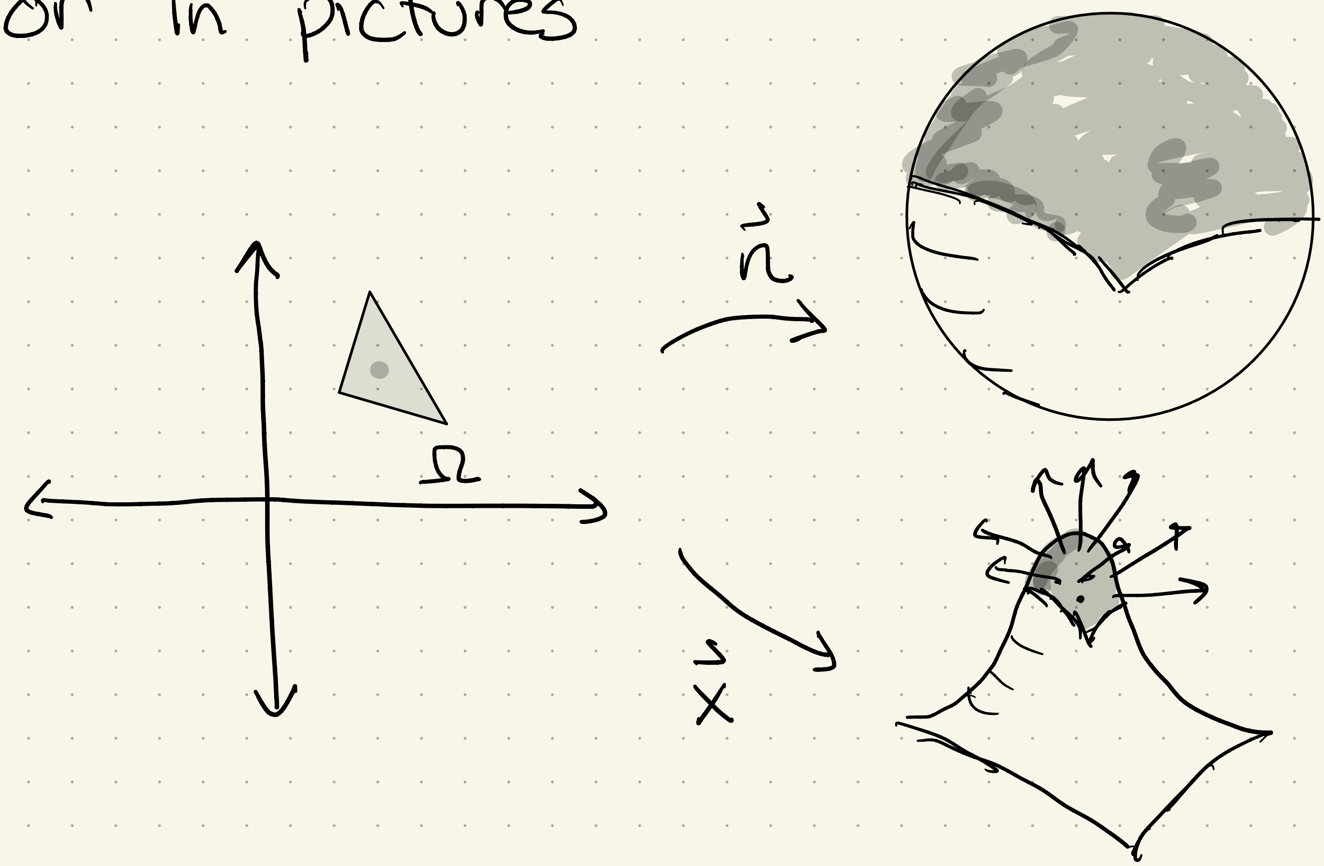
Theorem. The Gauss curvature is the pullback of the area form on S^2 by the Gauss map to the surface M.

(Oriented) area (on S^2) covered by normal vectors of the portion of the surface parametrized by $\Omega \subset \mathbb{R}^2$

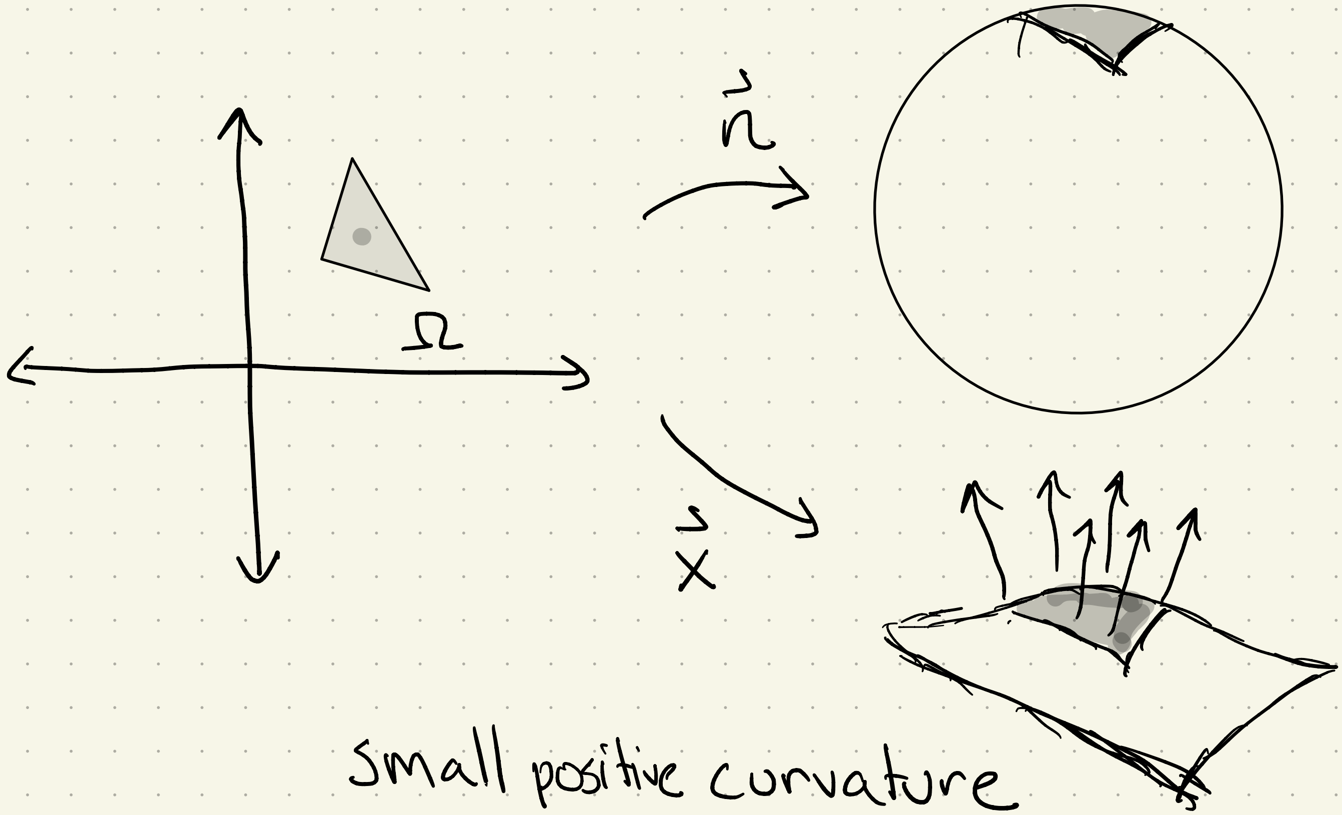
we integrate

$$\int_{\Omega} K(u,v) \underbrace{\sqrt{EG-F^2}\, du\, dv}_{\text{surface area on } M}$$

or in pictures



large positive curvature!



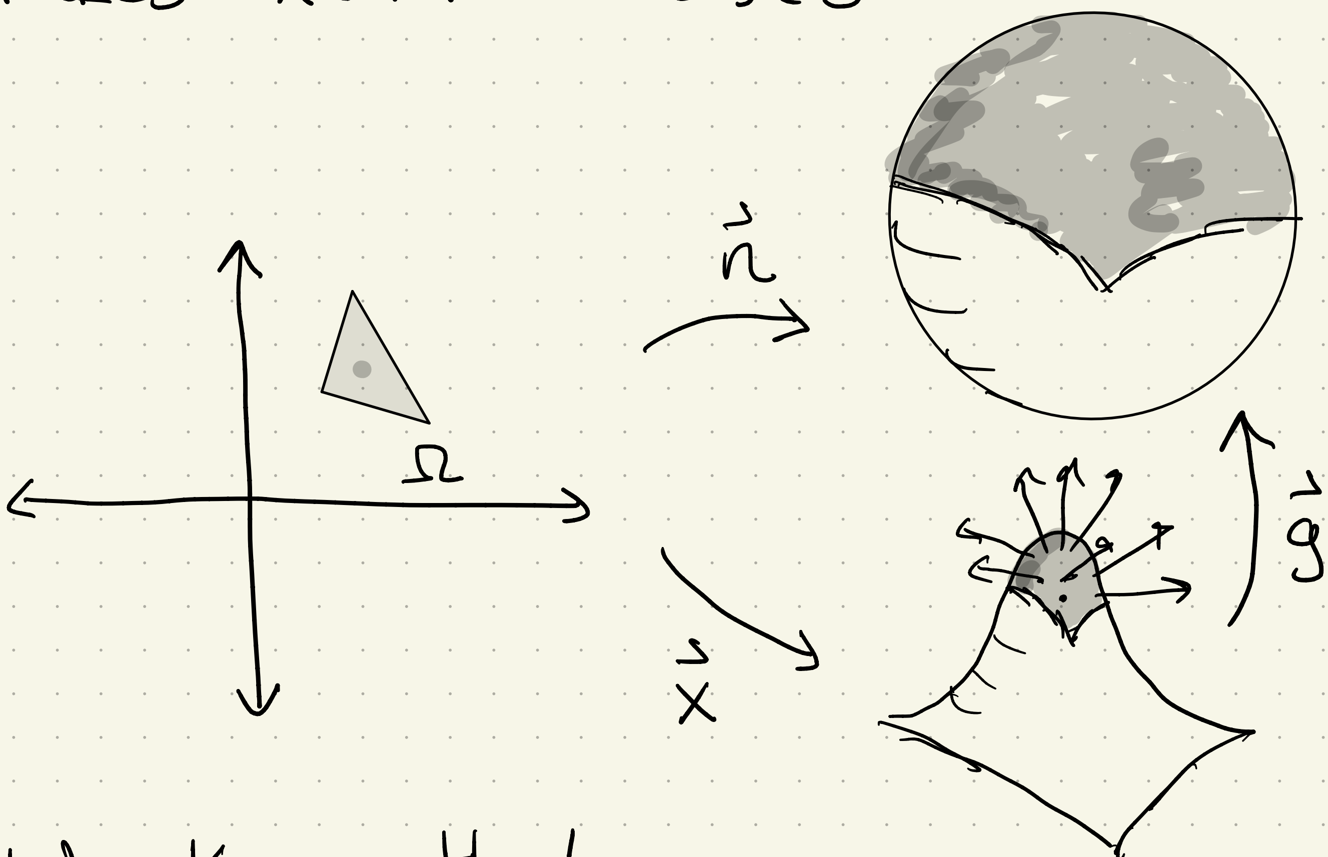
small positive curvature

A surface of negative curvature reverses orientation so the signed area of normals is negative.

Theorem. The total curvature of any compact surface without boundary

$\int_M K \, \sigma_M$ doesn't change when surface is smoothly deformed.

Proof. The gauss map $\vec{g}: M \to S^2$ takes $\vec{x} \in M \to \vec{n}(\vec{x}) \in S^2$



We know that

$$g^* \sigma_{S^2} = K \sigma_M$$

Suppose we have two surfaces M_0 and M_1 which can be deformed into each other.

Formally, this means there is a smooth map

$$M \times [0,1] \longrightarrow \mathbb{R}^3$$

so that M_0 = image of $(M,0)$ and M_1 = image of $(M,1)$. We can extend the gauss map to a map

$$\vec{g} : M \times [0,1] \longrightarrow S^2$$

by letting $\vec{g}(p,t)$ = normal vector to the surface at the image of p at time t.

Now

theorem about K

$$\int_{M_0} K\, d\sigma_{M_0} - \int_{M_1} K\, d\sigma_{M_1} = \int_{\partial(M\times[0,1])} \vec{g}^{*}\sigma_{S^2}$$

$$= \int_{M\times[0,1]} d(\vec{g}^{*}\sigma_{S^2}) = \int_{M\times[0,1]} \vec{g}^{*}(d\sigma_{S^2})$$

Stokes theorem

d and pullback commute

$$= \int_{M\times[0,1]} \vec{g}^{*}(0) = 0.$$

$\blacksquare$

S^2 is 2-dimensional and $d\sigma_2$ is a 3-form

Definition. A smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ is called a <u>hypersurface</u>.

If M is a hypersurface, $T_{\vec{p}}M$ is an $(n-1)$-dimensional hyperplane, whose orthogonal complement $(T_{\vec{p}}M)^{\perp}$ is 1-dimensional. A continuous choice of basis defines a Gauss map $\vec{g}: M \to S^{n-1}$, and a <u>Gauss</u> (or <u>scalar</u>) curvature K so that

$$\vec{g}^* \sigma_{S^{n-1}} = K \sigma_M$$

Theorem. Total curvature $\int_M K \sigma_M$ does not change when the hypersurface M is smoothly deformed.

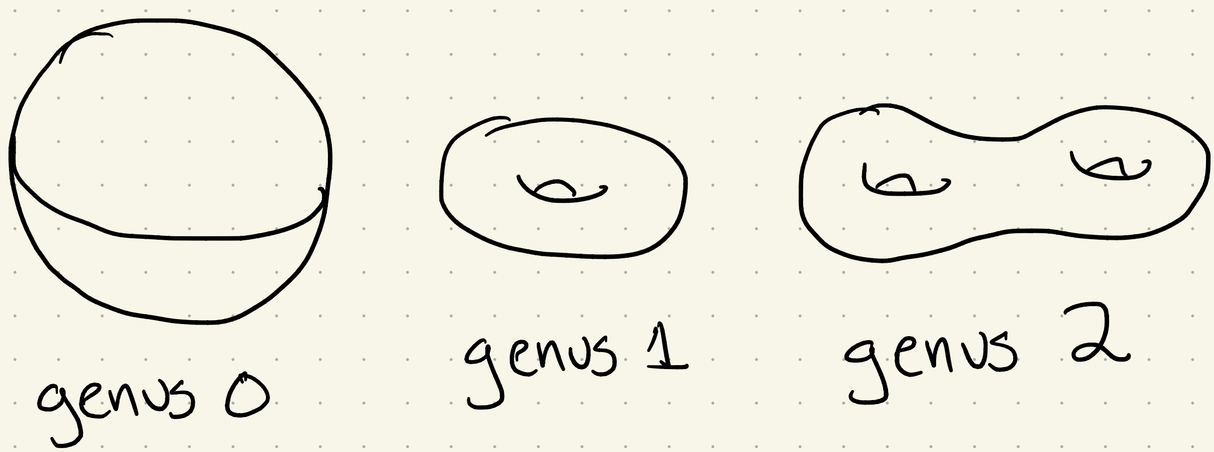
In fact, even more is true.

Theorem. (Gauss-Bonnet Theorem)

If M is a compact surface in $\mathbb{R}^3$ with no boundary or self-intersections,

$$\int K \, d\sigma_M = 2\pi(2-2g)$$

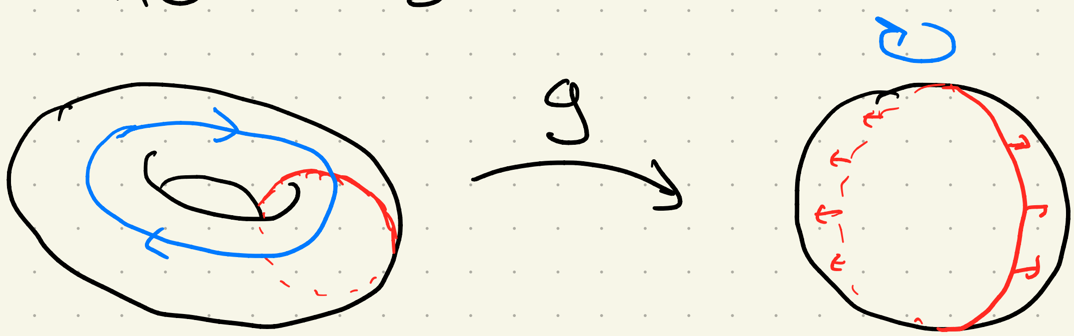
where g is the genus of the surface.



Proof idea. Every surface can be deformed to one of the models above.

For the sphere, $\vec{g}$ is the identity and $\int K \, d\sigma_M = \text{area} = 4\pi$.

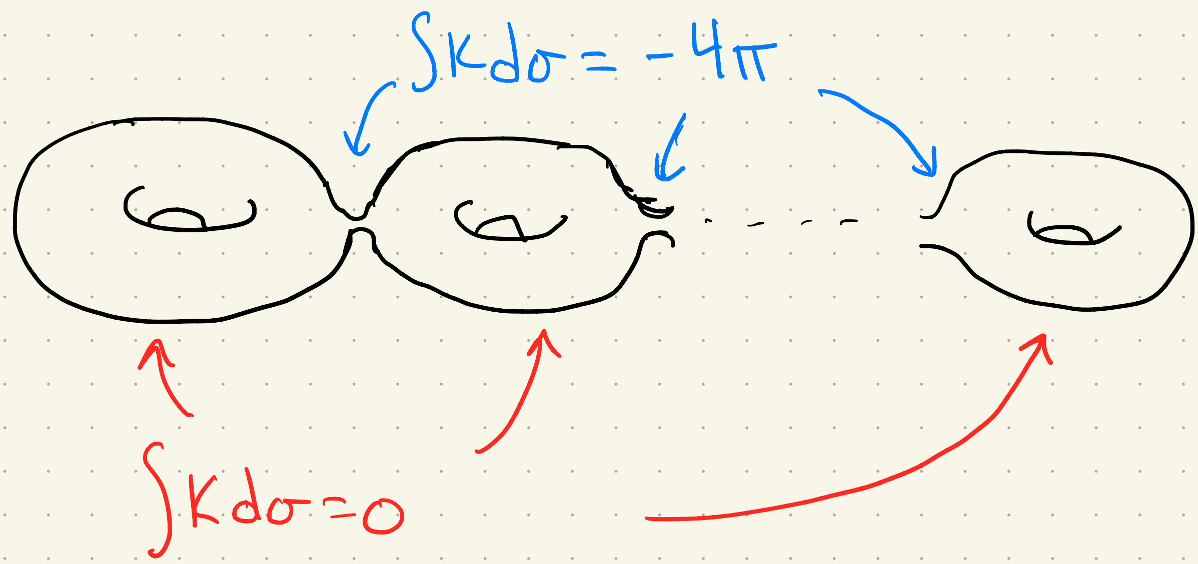
For the torus



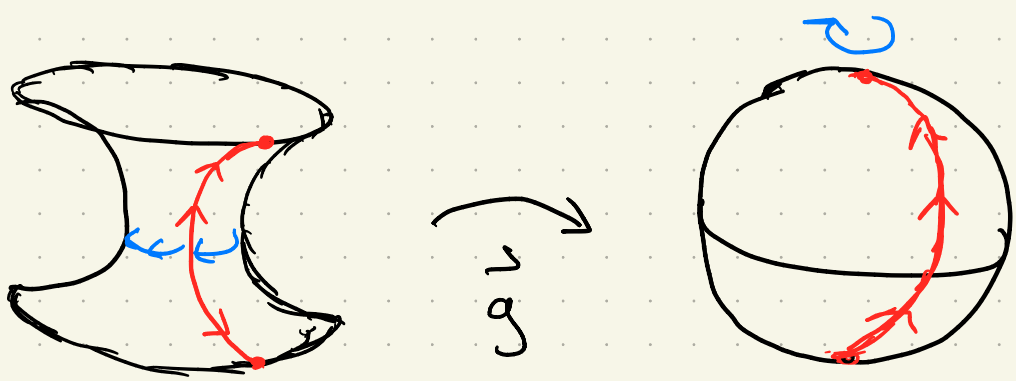
area is being swept out positively on one semicircle and negatively on the other at equal rates.

$$\int K \, d\sigma_M = \text{area} = 0$$

For the g-holed torus, we can deform the surface to



where the tiny "neck" is approximately

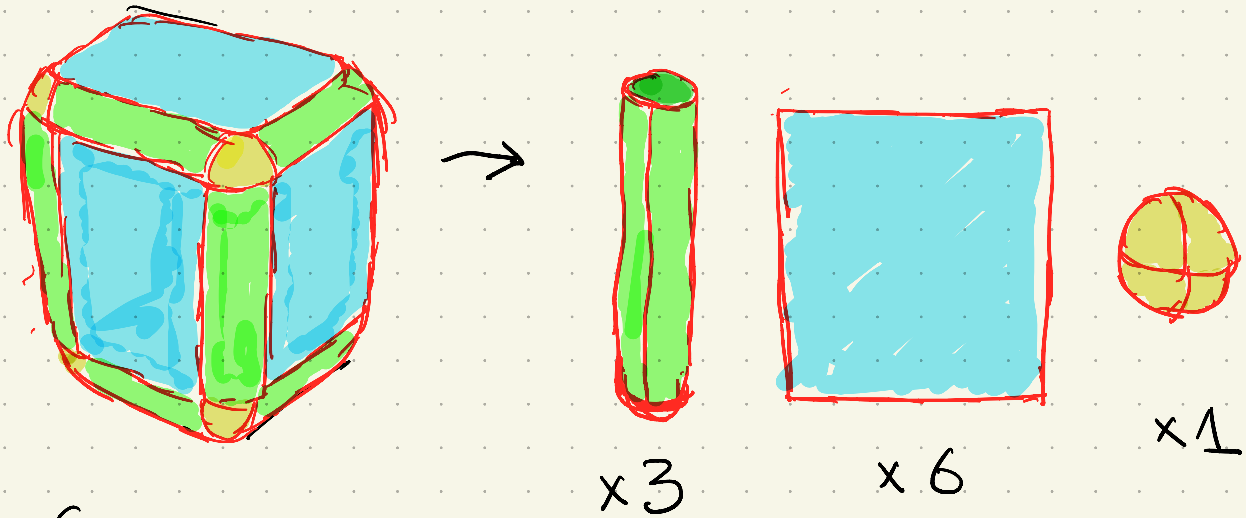


sweeps once over the sphere, but negatively oriented!

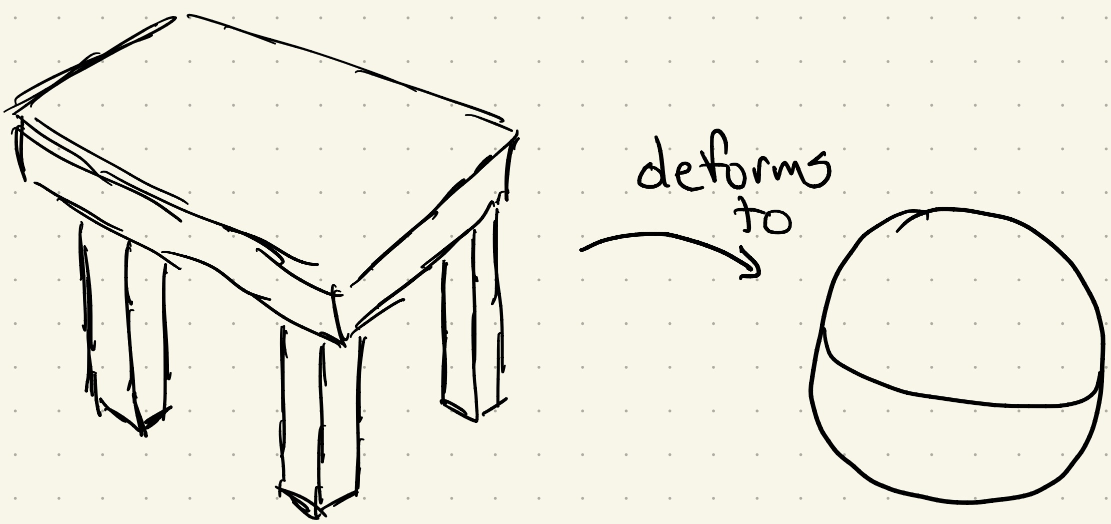
Counting $(g-1)$ necks:

$$\int K\, d\sigma = -4\pi(g-1) = 2\pi(2-2g). \qquad \blacksquare$$

Examples.

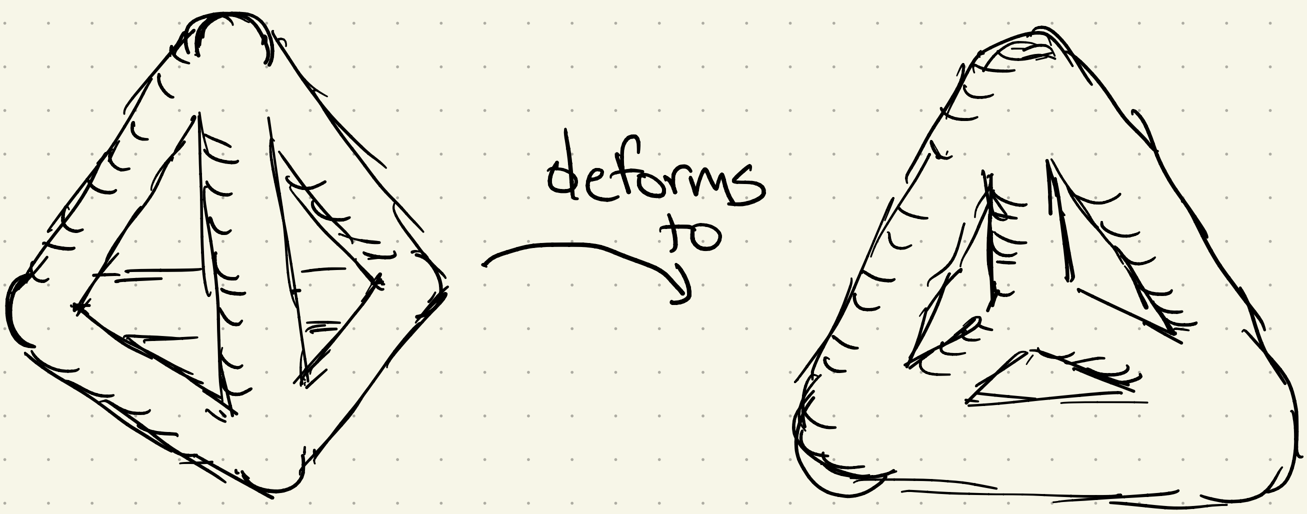


$$\underbrace{\int K\,d\sigma}_{=4\pi} = \underbrace{\int K\,d\sigma}_{=0} + \underbrace{\int K\,d\sigma}_{=0} + \underbrace{\int K\,d\sigma}_{=4\pi}$$



$$\int K\,d\sigma = 4\pi$$

$$\int K \, d\sigma = 0.$$



$$\int K \, d\sigma = -8\pi$$