The meaning of H

Suppose Ω is a region in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ and

$$\vec{X}(x,y) = (x, y, z(x,y))$$

We have previously computed

$$\vec{X}_x = (1, 0, z_x)$$

$$\vec{X}_y = (0, 1, z_y)$$

so

$$E = 1 + z_x^2, \quad F = z_x z_y, \quad G = 1 + z_y^2$$

Further,

$$\vec{X}_x \times \vec{X}_y = (-z_x, -z_y, 1)$$

so

$$\vec{n} = \frac{(-z_x, -z_y, 1)}{\sqrt{1 + z_x^2 + z_y^2}}$$

This means

$$\ell = \langle \vec{x}_{xx}, \vec{n} \rangle = \frac{z_{xx}}{\sqrt{1 + z_x^2 + z_y^2}}$$

$$m = \langle \vec{x}_{xy}, \vec{n} \rangle = \frac{z_{xy}}{\sqrt{1 + z_x^2 + z_y^2}}$$

$$n = \langle \vec{x}_{yy}, \vec{n} \rangle = \frac{z_{yy}}{\sqrt{1 + z_x^2 + z_y^2}}$$

We want to compute the trace of the shape operator. Since

$$S_p = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} \ell & m \\ m & n \end{bmatrix}$$

we can compute

$$S_p = \frac{1}{EG - F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} \ell & m \\ m & n \end{bmatrix}$$

$$= \frac{1}{EG - F^2} \begin{bmatrix} G\ell - Fm & \sim \\ \sim & -Fm + En \end{bmatrix}$$

so

$$\operatorname{tr} S_p = \frac{G\ell - 2Fm + En}{EG - F^2}$$

and

$$H = \frac{1}{2}\operatorname{tr} S_p = \frac{G\ell - 2Fm + En}{2(EG - F^2)}$$

Now

$$EG - F^2 = (1 + z_x^2)(1 + z_y^2) - z_x^2 z_y^2$$

$$= 1 + z_x^2 + z_y^2$$

In the numerator, we get

$$G\ell - 2Fm + En =$$

$$\frac{(1 + z_y^2) z_{xx} - 2 z_x z_y z_{xy} + (1 + z_x^2) z_{yy}}{\sqrt{1 + z_x^2 + z_y^2}}$$

Theorem. For a graph surface $(x, y, z(x,y))$

$$H = \frac{(1+z_y^2)z_{xx} - 2z_x z_y z_{xy} + (1+z_x^2)z_{yy}}{2(1+z_x^2+z_y^2)^{3/2}}$$

We now want to interpret this!

$$\begin{aligned} \text{Area}(z) &= \int_\Omega \sqrt{EG-F^2}\,dx\,dy \\ &= \int_\Omega \sqrt{1+z_x^2+z_y^2}\,dx\,dy \\ &= \int_\Omega \sqrt{1+\|\nabla z\|^2}\,dx\,dy \end{aligned}$$

Now suppose we want to compute the directional derivative

$$D_v \,\text{Area}(z) = \frac{d}{dt}\text{Area}(z+tv)$$

for some variation $v : \Omega \to \mathbb{R}$ with $v \equiv 0$ on $\partial\Omega$.

We can write down

$$\frac{d}{dt}\text{Area}(z+tv)\Big|_{t=0} =$$

$$= \int_\Omega \frac{d}{dt}\sqrt{1+(z+tv)_x^2+(z+tv)_y^2}\,\Big|_{t=0} dxdy$$

Now

$$\frac{d}{dt}(z+tv)_x^2\Big|_{t=0} = \frac{d}{dt}(z_x+tv_x)^2\Big|_{t=0}$$

$$= 2(z_x+tv_x)\cdot v_x\Big|_{t=0}$$

$$= 2z_xv_x$$

Similarly,

$$\frac{d}{dt}(z+tv)_y\Big|_{t=0} = 2z_yv_y.$$

Therefore

$$\frac{d}{dt}\sqrt{1+(z+tv)_x^2+(z+tv)_y^2}\,\Big|_{t=0} =$$

$$= \frac{1}{\cancel{2}\sqrt{1+z_x^2+z_y^2}} \cdot \left(\cancel{2} z_x v_x + \cancel{2} z_y v_y\right)$$

$$= \left\langle \frac{\nabla z}{\sqrt{1+\|\nabla z\|^2}}, \nabla v \right\rangle$$

Now in vector calculus, we learn to define the divergence of a vector field $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x_1} F_1 + \frac{\partial}{\partial x_2} F_2$$

Thus, for any scalar function φ,

$$\operatorname{div} \varphi\vec{F} = \frac{\partial}{\partial x_1} \varphi F_1 + \frac{\partial}{\partial x_2} \varphi F_2$$

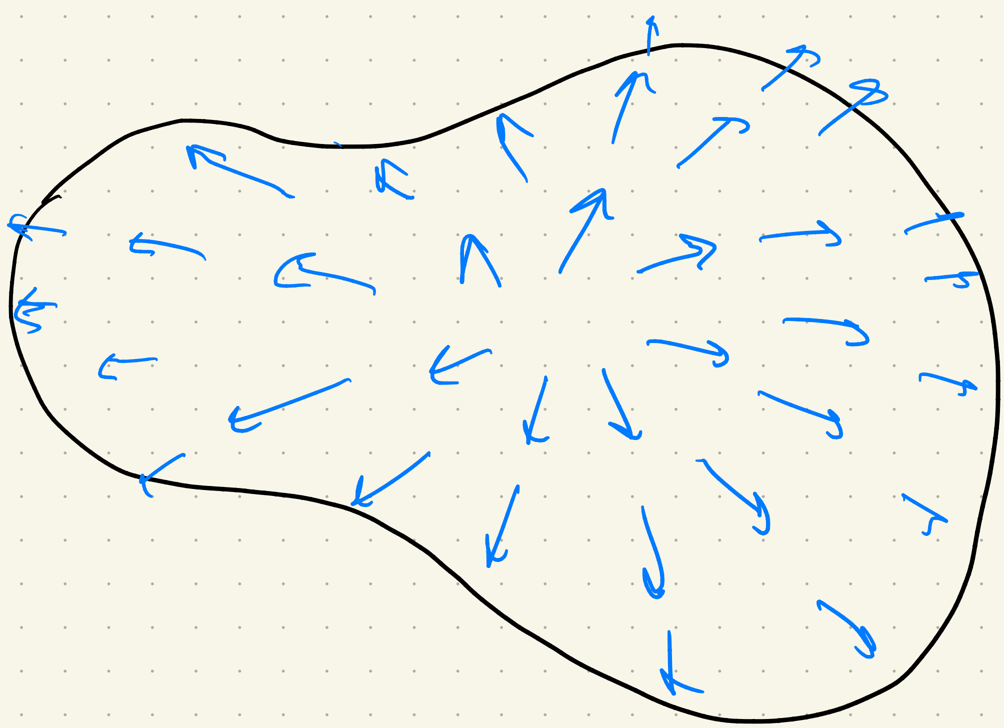
$$= \left(\frac{\partial}{\partial x_1}\varphi\right)F_1 + \varphi\left(\frac{\partial}{\partial x_1}F_1\right) + \left(\frac{\partial}{\partial x_2}\varphi\right)F_2 + \varphi\left(\frac{\partial}{\partial x_2}F_2\right)$$

$$= \langle \nabla\varphi, F\rangle + \varphi \operatorname{div} F.$$

We also learn the <u>divergence theorem</u>,

$$\int_{\Omega} \operatorname{div} F \, dArea = \int_{\partial\Omega} \langle F, \vec{n}\rangle \, ds$$

where $\vec{n}$ is the outward pointing unit normal to $\partial\Omega$.



Putting these together,

$$\int_{\Omega} \left\langle \frac{\nabla z}{\sqrt{1+\|\nabla z\|^2}}, \nabla v \right\rangle d\text{Area} =$$

$$= \int_{\Omega} \operatorname{div}\left(v \frac{\nabla z}{\sqrt{1+\|\nabla z\|^2}} \right) - v \operatorname{div}\left(\frac{\nabla z}{\sqrt{1+\|\nabla z\|^2}} \right) d\text{Area}$$

$$= \underbrace{\cancel{\int_{\partial\Omega} \left\langle v \frac{\nabla z}{\sqrt{1+\|\nabla z\|^2}}, \vec{n} \right\rangle ds}}_{0,\ \text{because } v \equiv 0 \text{ on } \partial\Omega} - \int_{\Omega} v \operatorname{div}\left(\frac{\nabla z}{\sqrt{1+|\nabla z|^2}} \right) d\text{Area}$$

So $D_v \text{Area}(z) = 0$ for all variations v if and only if $\operatorname{div}\left(\frac{\nabla z}{\sqrt{1+|\nabla z|^2}} \right) \equiv 0$.

So now let's compute

$$\operatorname{div}\left(\frac{1}{\sqrt{1+z_x^2+z_y^2}}\nabla z\right) = \left\langle \nabla\left(\frac{1}{\sqrt{1+z_x^2+z_y^2}}\right), \nabla z \right\rangle + \frac{z_{xx}+z_{yy}}{\sqrt{1+z_x^2+z_y^2}}$$

Now

$$\nabla\left(\frac{1}{\sqrt{1+z_x^2+z_y^2}}\right) = -\frac{1}{\cancel{2}}\frac{1}{(1+z_x^2+z_y^2)^{3/2}}\begin{bmatrix} \cancel{2}z_xz_{xx}+\cancel{2}z_yz_{yx} \\ \cancel{2}z_xz_{xy}+\cancel{2}z_yz_{yy} \end{bmatrix}$$

so

$$\left\langle \nabla\frac{1}{\sqrt{1+z_x^2+z_y^2}}, \nabla z \right\rangle + \frac{z_{xx}+z_{yy}}{\sqrt{1+z_x^2+z_y^2}}$$

$$= \frac{-z_x^2z_{xx} - z_xz_yz_{yx} - z_xz_yz_{xy} - z_y^2z_{yy}}{(1+z_x^2+z_y^2)^{3/2}} + \frac{(1+z_x^2+z_y^2)(z_{xx}+z_{yy})}{(1+z_x^2+z_y^2)^{3/2}}$$

$$= \frac{(1+z_y^2)z_{xx} - 2z_x z_y z_{xy} + (1+z_x^2)z_{yy}}{(1+z_x^2+z_y^2)^{3/2}}$$

$$+ \frac{\cancel{z_x^2 z_{xx} - z_x^2 z_{xx}} + \cancel{z_y^2 z_{yy} - z_y^2 z_{yy}}}{(1+z_x^2+z_y^2)^{3/2}}$$

which proves a striking theorem!

Theorem. $2H$ is the first variation of area.

Surfaces with $H \equiv 0$ are area-critical

we call them <u>minimal</u> surfaces.

Example. $\left(x, y, \log\left(\frac{\cos x}{\cos y}\right)\right)$.

We compute

$$z(x,y) = \log(\cos x) - \log(\cos y)$$

so

$$z_x = \frac{1}{\cos x} \cdot (-\sin x) = -\tan x$$

$$z_y = \frac{-1}{\cos y}(-\sin y) = \tan y$$

and

$$z_{xx} = -\sec^2 x$$

$$z_{xy} = 0$$

$$z_{yy} = -\sec^2 y$$

so

$$H = \frac{(1+z_y^2)z_{xx} - 2z_xz_yz_{xy} + (1+z_x^2)z_{yy}}{2(1+z_x^2+z_y^2)^{3/2}}$$

$$= \frac{-(1+\tan^2 y)\sec^2 x + (1+\tan^2 x)\sec^2 y}{2(1+\tan^2 x+\tan^2 y)^{3/2}}$$

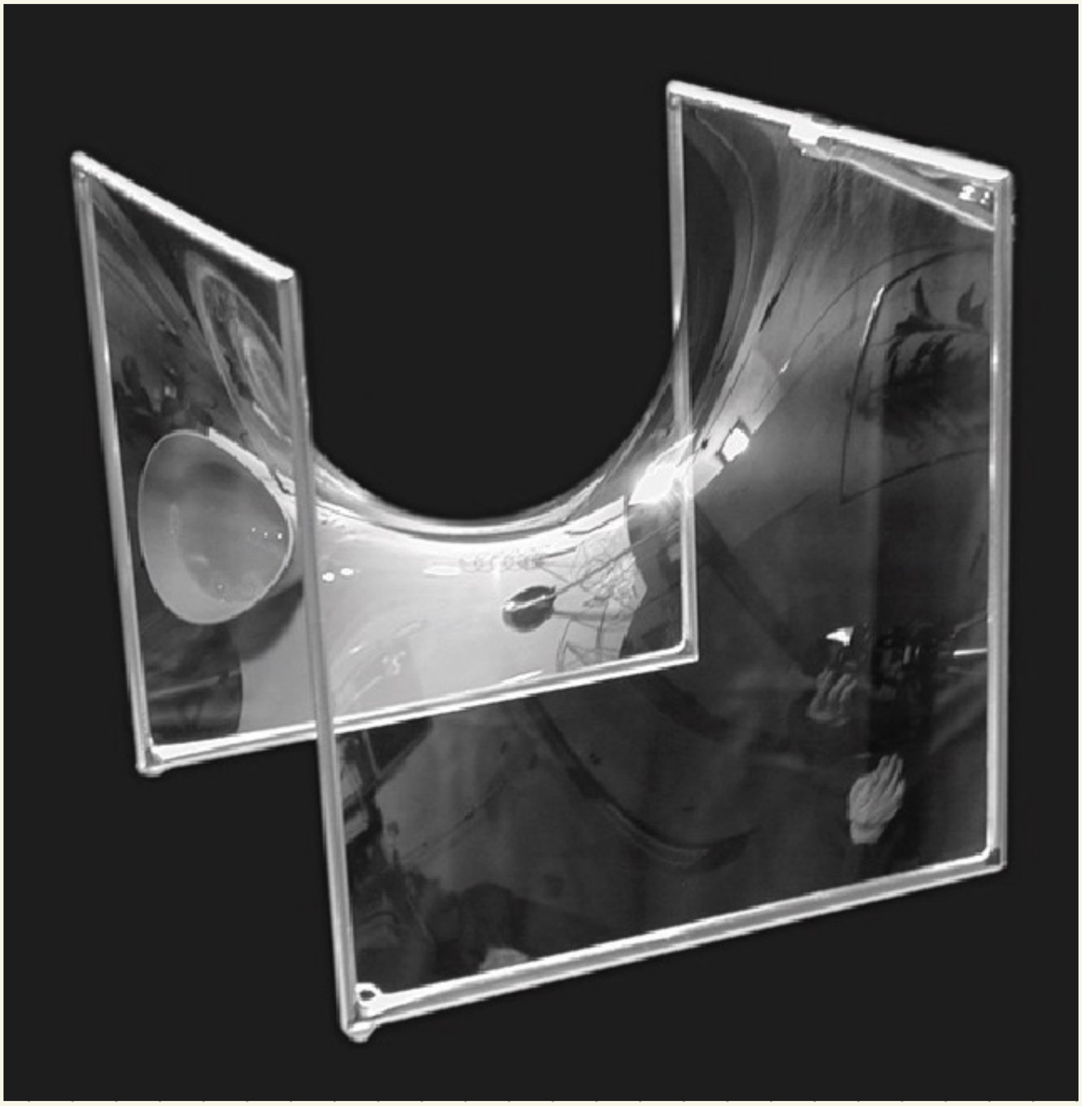
But $1+\tan^2\theta = \sec^2\theta$ for all θ, so

$$= \frac{-\sec^2 y\sec^2 x + \sec^2 x\sec^2 y}{2(1+\tan^2 x+\tan^2 y)^{3/2}}$$

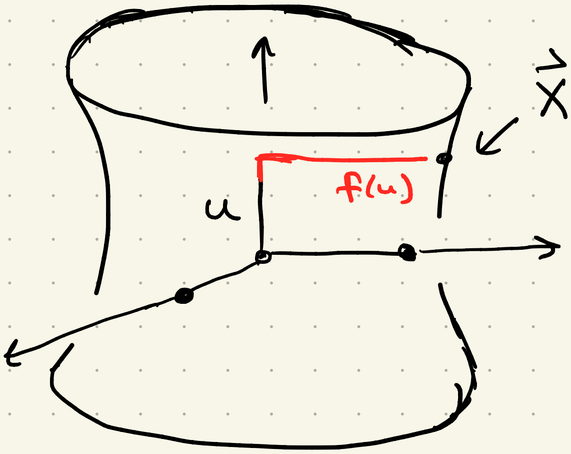
$$= 0$$

and this is well-defined as long as $x, y \in (-\pi/2, \pi/2)$ so $\cos x, \cos y \neq 0$.

This is called Scherk's surface.



Example. In previous notes, we considered the surface of revolution



$$\vec{x}(u,v) = (f(u)\cos v,\ f(u)\sin v,\ u)$$

and showed that

$$H = \frac{(1+(f')^2) - ff''}{2f(1+(f')^2)^{3/2}} \operatorname{sign}(f)$$

This means that $H = 0 \iff$

$$(1+(f')^2) - ff'' = 0$$

$$f'' = \frac{1+(f')^2}{f}$$

This is a 2nd order ODE for f.

We can solve it by guessing and checking.

$$f(u) = c \cosh \frac{u}{c}$$

$$f'(u) = \sinh \frac{u}{c}$$

$$f''(u) = \frac{1}{c} \cosh \frac{u}{c}$$

So

$$\frac{1 + f'(u)^2}{f(u)} = \frac{1 + \sinh^2 \frac{u}{c}}{c \cosh \frac{u}{c}} = \frac{\cosh^{\cancel{2}} \frac{u}{c}}{c \cancel{\cosh \frac{u}{c}}} = f''(u)$$

(In fact, I think this is the only real solution, as $f(u) = iu$ also solves the ODE.)

The resulting surface is the catenoid!