Maxwell's Equations

Math 3510

8.6.3

In the classical theory of electromagnetism there are 3 vector fields and 1 function.

Fields	Function
E = electric field	ρ = charge density
B = magnetic field	
J = current density	

These are related by four equations:

Gauss' law: $\operatorname{div} E = \rho$

(No magnetic monopoles): $\operatorname{div} B = 0$

Faraday's law: $\operatorname{curl} E = -\dfrac{\partial B}{\partial t}$

Ampère's law: $\operatorname{curl} B = \dfrac{\partial E}{\partial t} + J$

We now want to understand electricity and magnetism from our new perspective.

Spacetime:

$\mathbb{R}^4$ with coordinates x, y, z, t and volume form $dt \wedge dx \wedge dy \wedge dz$

We will need a new dictionary. Previously, we used

$$\Lambda^0(\mathbb{R}^3) \cong \mathbb{R} \qquad \dim \binom{3}{0} = 1$$

$$\Lambda^1(\mathbb{R}^3) \cong \mathbb{R}^3 \qquad \dim \binom{3}{1} = 3$$

$$\Lambda^2(\mathbb{R}^3) \cong \mathbb{R}^3 \qquad \dim \binom{3}{2} = 3$$

$$\Lambda^3(\mathbb{R}^3) \cong \mathbb{R} \qquad \dim \binom{3}{3} = 1$$

In spacetime we'll focus on

$$\Lambda^1(\mathbb{R}^4) \cong \mathbb{R}^4 \quad \dim \binom{4}{1} = 4$$

$$\Lambda^2(\mathbb{R}^4) \cong \mathbb{R}^6 \quad \dim \binom{4}{2} = 6$$

$$\Lambda^3(\mathbb{R}^4) \cong \mathbb{R}^4 \quad \dim \binom{4}{3} = 4$$

We note that $\star : \Lambda^2(\mathbb{R}^4) \to \Lambda^2(\mathbb{R}^4)$ (and also $\star : \Lambda^1(\mathbb{R}^4) \to \Lambda^3(\mathbb{R}^4)$)

We will need a new dictionary to understand the relationship between forms, fields, and functions.

2-forms "Bivector" fields

$$\begin{aligned} &v_1\, dx \wedge dt \quad && \omega_1\, dy \wedge dz \\ &v_2\, dy \wedge dt \; + && \omega_2\, dz \wedge dx \\ &v_3\, dz \wedge dt && \omega_3\, dx \wedge dy \end{aligned} \qquad \begin{aligned} &\vec{V}(\vec{x}, t) \\ &\vec{W}(\vec{x}, t) \end{aligned}$$

Hodge star (modified for Lorentz metric)

$$\star \begin{pmatrix} v_1\, dx\wedge dt & \omega_1\, dy\wedge dz \\ v_2\, dy\wedge dt + & \omega_2\, dz\wedge dx \\ v_3\, dz\wedge dt & \omega_3\, dx\wedge dy \end{pmatrix} = \begin{matrix} \omega_1\, dx\wedge dt - v_1\, dy\wedge dz \\ \omega_2\, dy\wedge dt - v_2\, dz\wedge dx \\ \omega_3\, dz\wedge dt - v_3\, dx\wedge dy \end{matrix}$$

or

$$\star(\vec{V}(\vec{x},t), \vec{W}(\vec{x},t)) = (W(\vec{x},t), -V(\vec{x},t))$$

3-forms

field + function

$$\omega = \begin{matrix} v_1\, dy\wedge dz\wedge dt \\ + v_2\, dz\wedge dx\wedge dt \\ + v_3\, dx\wedge dy\wedge dt \\ + f\, dx\wedge dy\wedge dz \end{matrix}$$

$$(\vec{V}(\vec{x},t), f(\vec{x},t))$$

1-forms

field + function

$$\omega = \begin{matrix} v_1\, dx \\ + v_2\, dy \\ + v_3\, dz \\ + f\, dt \end{matrix}$$

$$(\vec{V}(\vec{x},t), f(\vec{x},t))$$

While

$$\star \begin{pmatrix} V_1\, dy \wedge dz \wedge dt \\ + V_2\, dz \wedge dx \wedge dt \\ + V_3\, dx \wedge dy \wedge dt \\ + f\, dx \wedge dy \wedge dz \end{pmatrix} = \begin{matrix} -V_1\, dx \\ -V_2\, dy \\ -V_3\, dz \\ +f\, dt \end{matrix}$$

or

$$\star\left(\vec{V}(\vec{x}), f(\vec{x})\right) = \left(-\vec{V}(\vec{x}), f(\vec{x})\right).$$

Now d is Kind of interesting:

$$d\begin{pmatrix} V_1\, dx \wedge dt & & \omega_1\, dy \wedge dz \\ V_2\, dy \wedge dt & + & \omega_2\, dz \wedge dx \\ V_3\, dz \wedge dt & & \omega_3\, dx \wedge dy \end{pmatrix} =$$

$$\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} + \frac{\partial \omega_1}{\partial t}\right) dy \wedge dz \wedge dt$$

$$+ \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} + \frac{\partial \omega_2}{\partial t}\right) dz \wedge dx \wedge dt$$

$$+ \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} + \frac{\partial \omega_3}{\partial t}\right) dx \wedge dy \wedge dt$$

$$+\left(\frac{\partial w_1}{\partial x}+\frac{\partial w_2}{\partial y}+\frac{\partial w_3}{\partial z}\right) dx\wedge dy\wedge dz$$

or

$$d(\vec{V}(\vec{x},t),\vec{W}(\vec{x},\vec{t}))$$
$$=\left(\nabla\times\vec{V}(\vec{x},t)+\frac{\partial}{\partial t}\vec{W}(\vec{x},\vec{t}),\nabla\cdot\vec{W}(\vec{x},t)\right)$$

While

$$d\left(\begin{array}{l} v_1\, dy\wedge dz\wedge dt \\ +v_2\, dz\wedge dx\wedge dt \\ +v_3\, dx\wedge dy\wedge dt \\ +f\, dx\wedge dy\wedge dz \end{array}\right)=$$

$$\left(\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}-\frac{\partial f}{\partial t}\right) dx\wedge dy\wedge dz\wedge dt$$

or

$$d(\vec{V}(\vec{x},t),f(\vec{x},t))=\nabla\cdot V-\frac{\partial f}{\partial t}$$

In classical physics, we had the electric field $\vec{E}(\vec{x},t)$ and the magnetic field $\vec{B}(\vec{x},t)$. Together, these are the "electromagnetic 2-form" ω.

$$\omega \longleftrightarrow (\vec{E}(\vec{x},t), \vec{B}(\vec{x},t))$$

The first equation is

$$d\omega = 0 \longleftrightarrow \nabla \times E + \frac{d}{dt} B = 0 \quad \text{(Faraday's Law)}$$

$$\nabla \cdot B = 0 \quad \text{(no magnetic monopoles)}$$

The second object is the "current charge" 3-form, which describes the motion and density of electrons

$$\varphi \longleftrightarrow \big(\underbrace{\vec{J}(\vec{x},t)}_{\text{current flow}},\ -\underbrace{\rho(\vec{x},t)}_{\text{charge density}}\big)$$

The second equation is

$$d(\star\omega) = \varphi \leftrightarrow d\big(\vec{B}(\vec{x},t), -\vec{E}(\vec{x},t)\big)$$

$$\leftrightarrow \nabla\times\vec{B} - \frac{\partial}{\partial t}\vec{E} = \vec{J} \quad \text{(Ampère's Law)}$$

$$-\nabla\cdot E = -\rho \quad \text{(Gauss' Law)}$$

Now "to solve Maxwell's equations" means, basically,

"Given current flow (J) and charge density (ρ), find the corresponding electric and magnetic fields (E, J)."

or, in our new language,

"Given the 3-form Θ, find a 2-form ω so that $d(\star\omega) = \Theta$ and $d\omega = 0$."

We will now outline how this may be done.

Since $d\omega = 0$ on all of $\mathbb{R}^4$, $\exists$ some 1-form α so $d\alpha = \omega$.

Now α is not unique, since

$$d(\alpha + df) = d\alpha + \underbrace{d(df)}^{0} = \omega$$

for any 0-form f. So let's choose f craftily by solving the PDE

$$(d\star d)f = -d\star\alpha$$

and let $\beta = \alpha + df$. Then

$$d\beta = \omega, \quad d\star\beta = d\star\alpha + d\star df = d\star\alpha - d\star\alpha = 0.$$

This trick is called "choice of gauge".

Now suppose

$$\beta = A_1 dx + A_2 dy + A_3 dz + \varphi dt$$

Then

$$\begin{aligned} \star\beta = \quad & A_1 \, dy \wedge dz \wedge dt \\ & A_2 \, dz \wedge dx \wedge dt \\ & A_3 \, dx \wedge dy \wedge dt \\ & -\varphi \, dx \wedge dy \wedge dz \end{aligned}$$

and

$$0 = d\star\beta =$$

$$= \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} + \frac{\partial \varphi}{\partial t} \right) dx \wedge dy \wedge dz \wedge dt$$

$$\Rightarrow \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} + \frac{\partial \varphi}{\partial t} \right) = 0$$

Further $d\beta = \omega$, so

$$(d\star d)\beta = d\star\omega$$
$$= \theta$$

So we have learned that we may define β to be the unique 1-form so that

$$(d\star d)\beta = \theta, \quad d\star\beta = 0.$$

and then obtain ω by computing $d\beta$. In more familiar terms, these equations become

$$\left.\begin{aligned} \Delta A - \frac{\partial^2 A}{\partial t^2} &= -J \\ \Delta \varphi - \frac{\partial^2 \varphi}{\partial t^2} &= -\rho \end{aligned}\right\} \text{wave equations}$$

The field A is called the "vector potential" while φ is called the "scalar potential".